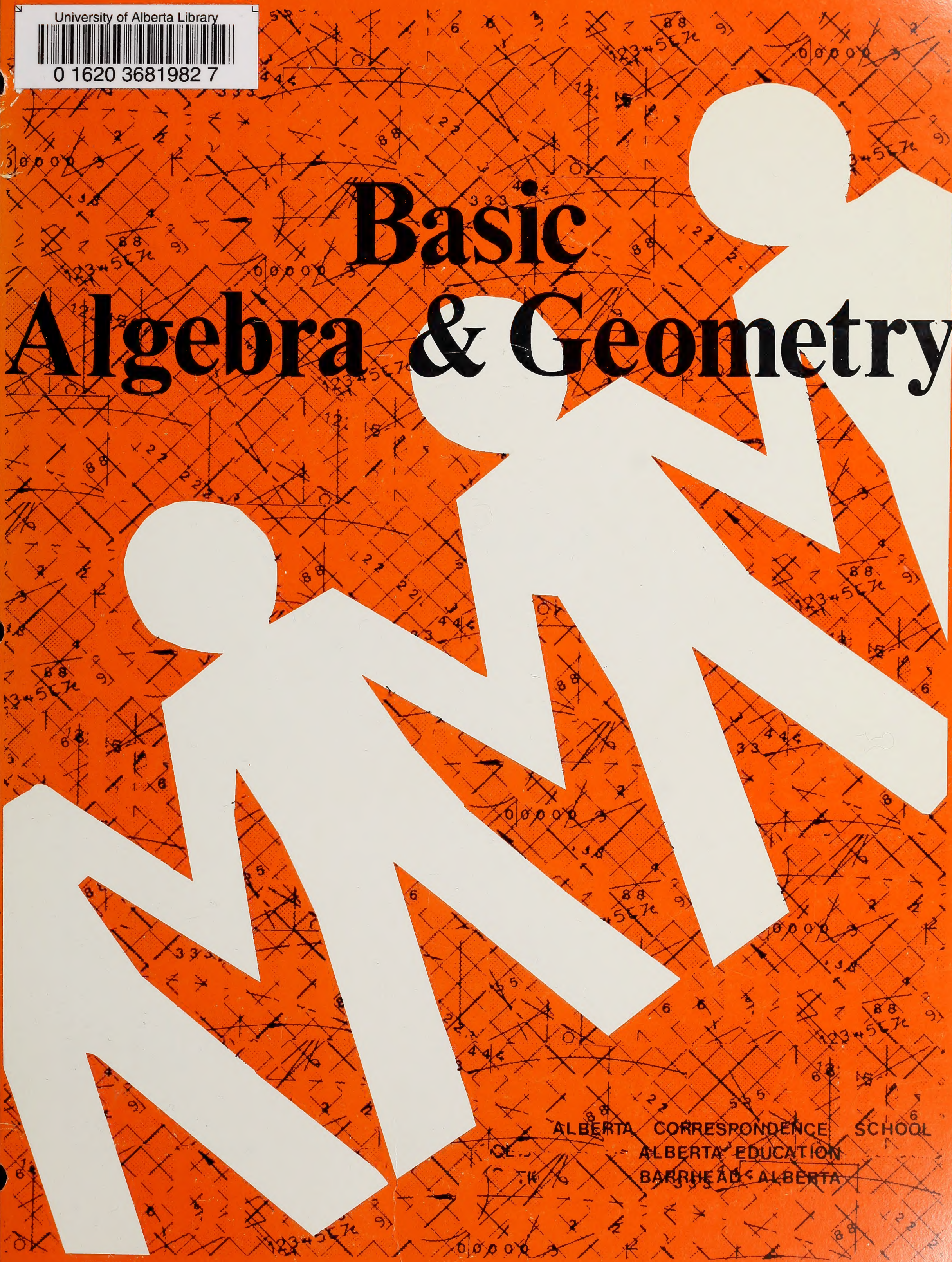
Basic Algebra & Geometry



ALBERTA CORRESPONDENCE SCHOOL
ALBERTA EDUCATION
BARRHEAD ALBERTA

Basic Algebra and Geometry

Lessons 1-10

Distance Learning

Alberta
EDUCATION

Basic Algebra and Geometry
Student Module
Lessons 1-10 and Answers
Alberta Correspondence School
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IMPORTANT!

PLEASE READ THIS INTRODUCTION BEFORE BEGINNING LESSON 1.

Content and Purpose of Course

Basic Algebra and Geometry is a ten-lesson, non-credit course which covers important topics from the present junior high school mathematics curriculum. It provides background material for students wishing to enrol in Mathematics 10 or Mathematics 13 at the high school level. This course is also designed as a lead-in to Mathematics Upgrading, a special adult course which prepares students for the course, Mathematics 30. Only students who obtain very good gradings in Basic Algebra and Geometry are advised to make the jump to Mathematics Upgrading.

The following list outlines the major topics covered in this course.

Lesson	Topics
1	Sets; Natural and Whole Numbers
2	Properties of Whole Numbers
3	Integers
4	Rational Numbers
5	Decimal Numbers
6	Real Numbers
7	Powers
8	Variable Expressions
9	Using Algebra
10	Geometry

Textbooks

There is no prescribed textbook that accompanies the lessons in this course. All the explanatory material is provided in the lessons. If you would like additional material on certain topics, the recommended textbooks for Grade 9 Mathematics are a good source of reference material. These textbooks are Contemporary Mathematics 3 by J.E. Dean et al, Exploring Modern Mathematics, Book 3 by M.L. Keedy et al; and Mathematics-Concepts, Applications: Second Course by H. Van Engen et al.

Self-correcting Exercises

A number of self-correcting exercises appear in the lessons in this course. You must work through each of these exercises carefully and when you have completed the entire exercise, check your answers with those provided at the end of the lesson. Correct any errors that you have made and ask about any of the answers that you do not understand. It is important that you handle these self-correcting exercises properly as they help prepare you for the assignments that follow.

General Instructions

1. Remove the staples from your set of lessons and transfer the lesson pages to a looseleaf binder which will preserve both the unused and corrected lessons. Do not tear pages from the book.

2. Do the exercises and lessons in their proper order.

3. When working on a lesson, do not skip any reading material. Each lesson requires careful reading, and on many pages you must fill in a number of blanks. Make sure that you fill in all these blanks.

4. Memorize important definitions, rules, and procedures from each lesson before you proceed to the next lesson. This will help you retain the material for use in future courses.

In order to complete Lesson 10 in this course you will need a ruler and protractor. Obtain these items before you begin that lesson.

This upgrading course is a self-study course in which you will correct your own answers. These answers are given in the Booklet of Solutions and Answers. Appropriate explanations are also supplied. **Please do not submit these lessons to the Alberta Correspodnence School for correction.**

In order for effective learning to occur, study the lesson notes very carefully so that the concepts are understood. Once the material is fully comprehended, complete the questions, and then check the solutions and answers in the booklet provided. Any questions which you found difficult or could not do should be carefully noted. Study the answers provided, and also review the appropriate background notes in the lesson.

If additional assistance is needed, you may telephone the Alberta Correspondence School at 674-5333. Students who reside in Alberta may call the school free of charge through the local RITE number in your area or the Zenith number 22333, if there is no RITE centre in your calling area.

We wish you success in your efforts.

Lesson 1

Sets; Natural and Whole Numbers

Basic Algebra and Geometry

SETS; NATURAL AND WHOLE NUMBERS

We are going to begin this course with a discussion of set theory. If you have been out of school for a number of years, you probably haven't heard the word "set" used in mathematics before. In recent years, set theory has become very important in the teaching of arithmetic, algebra, and geometry. In this lesson, you will be introduced to the terms and notation used in mathematics when talking about sets.

Topic One: Sets

We often find it useful to talk about collections or groups of objects. For example, you have heard people refer to a set of golf clubs, a set of dishes, a set of books and so on.

A SET may be described as a well-defined collection of objects. By "well-defined" we mean that we are able to tell whether or not a certain object belongs to the set. Any object belonging to a particular set is called an ELEMENT of that set.

EXAMPLE: The set of provinces in Canada is a well-defined set since there is no confusion as to the elements of this set.

Is "Alberta" an element of this set? ________ Is "Idaho"? ________

Is "5"? ________ Is "Ontario"? ________ Is "Calgary"? ________

A. Tabulating Sets

We can describe a set by listing its elements within a set of curly braces, { }. Each member of the set is listed only once. Commas are used to separate the members. The elements of the set may be listed in any order. For convenience, we can choose some capital letter to represent the set. When we list the elements of a set in this manner, we call this TABULATING the set.

EXAMPLE: Tabulate set A, the set of letters in the word "collection".

Solution

The letters c, o, and l appear more than once in the word "collection" but should be listed only once in the tabulation.

$A = \{c, o, l, e, t, i, n\}$ USE CURLY BRACES AROUND THE LISTING.

USE COMMAS BETWEEN ELEMENTS.

This is read, "A is the set whose elements are c, o, l, e, t, i, and n." Since the order in which we list the elements is not important, this set could also be given as $\{l, o, e, c, n, i, t\}$, $\{e, n, t, i, l, o, c\}$ and so on.

Self-correcting Exercise #1

Answers to this exercise may be found on page 33 of this lesson.

1. Tabulate each of the following sets. (Remember to use curly braces around the sets and put commas between the elements.)

   (a) The set whose elements are 5, 30, and 95.

   ____

   (b) The set of days in the week.

   ____

   (c) The set of letters in the word "attention".

   ____

   (d) The set of digits in the number 1 103 350.

   ____

2. Tell how you would read each of these tabulations.

   (a) $C = \{8, 3, 2\}$

   ____

   (b) $R = \{a, b\}$

   ____

In Arithmetic, we deal mainly with sets of numbers. Sometimes when we are tabulating sets of numbers, it is either inconvenient or impossible to list all the elements in a given set. When the elements of such a set follow a regular sequence or pattern, we can use three dots "..." to indicate that some of the elements have been omitted in the tabulation. Thus we use the following procedure:

1. List the first 3 elements (putting a comma after each).
2. Write 3 dots followed by a comma to indicate that the next elements are omitted but follow the same pattern.
3. Give the last element in the sequence (if there is one).

EXAMPLE: Tabulate the set of even numbers between 4 and 100.
<u>Solution</u>

The first even number <u>between</u> 4 and 100 is 6. The second is 8, and the third is 10. The pattern continues until you arrive at the last number, 98.

$$\{\underbrace{6, 8, 10,}_{\text{FIRST THREE ELEMENTS (WITH A COMMA AFTER EACH)}}\ \underbrace{\ldots,}_{\text{3 DOTS (FOLLOWED BY A COMMA)}} 98\}$$

LAST ELEMENT

(Note that 4 and 100 do <u>not</u> belong to the set. The numbers in the set must fall "between" these boundaries.)

Two important sets of numbers that you will often encounter are the set of natural numbers, N, and the set of whole numbers, W. Note how these sets are tabulated.

THESE 3 DOTS ARE READ "AND SO ON".

$$N = \{1, 2, 3, \ldots\}$$

THIS SET CAN ALSO BE REFERRED TO AS THE SET OF COUNTING NUMBERS.

$$W = \{0, 1, 2, \ldots\}$$

Since both these sets go on indefinitely, no last element can be given in their tabulations. Sets N and W differ only in the one respect that zero belongs to set W but not to set N (i.e. zero is a whole number but not a natural number).

<u>Self-correcting Exercise #2</u>

Answers to this exercise may be found on page 33 of this lesson.

1. Tabulate each of the following sets, using three dots to indicate omitted members.

   (a) The set of all natural numbers less than 130.

   ____

   (b) The set of all whole numbers less than 130.

   ____

   (c) The set of natural numbers greater than 25.

   ____

   (d) The set of multiples of 6 between 12 and 150.

   ____

(e) The set of counting numbers greater than or equal to 5 and less than or equal to 200.

____

2. M = {5, 10, 15, ..., 95}
Decide whether or not each element listed below belongs to set M. (Write "yes" if it does and "no" if it doesn't.)

(a) 5 _____	(b) 6 _____	(c) 12 _____	(d) 16 _____
(e) 20 _____	(f) 29 _____	(g) 30 _____	(h) 92 _____
(i) 65 _____	(j) 96 _____	(k) 100 _____	(l) 0 _____

B. The Number of Elements in a Set

When we can count the number of elements in a set, we say that the set is FINITE.

For example, the set of months in a year is a finite set since it has 12 members which can be counted. The set {1, 2, 3, ..., 100} is also finite since it has a limited number of elements (100 to be exact). The number of elements in a finite set is called its CARDINAL NUMBER. We can determine this number by a counting process. We set up a correspondence between the elements in the set and the set of natural numbers. When the count ends, we know that the last number named tells us the number of elements in the set.

EXAMPLE: What is the cardinal number of the set of letters in the word "green"?
Solution

This set may be tabulated as follows:
{g, r, e, n}
Now, count the elements.

$$\begin{array}{cccc} g, & r, & e, & n \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 & 4 \end{array}$$

The cardinal number of this set is 4.

If a set has an unlimited number of elements, it is called an INFINITE SET.

There is no end to listing the elements of an infinite set. For example, the set of all natural numbers N = {1, 2, 3, ...} is an infinite set. Set N is tabulated with three dots at the end to indicate that the same pattern continues, but no last member can be listed.

A set which contains no members is called an EMPTY SET and is referred to as the NULL SET.

The null set is represented by a pair of empty braces, { }, or by the Greek letter phi, ∅. The set of counting numbers that are greater than 9 but less than 10 contains no members. Thus, it is an empty set and can be represented by the symbol, { }, or the symbol, ∅.

i.e. $\left\{\begin{array}{c}\text{All counting numbers greater} \\ \text{than 9 but less than 10}\end{array}\right\} = \{\ \}$ or $\emptyset$.

Self-correcting Exercise #3

Answers to this exercise may be found on page 33 of this lesson.

1. Give the cardinal number of each of these sets.

   (a) {3, 8, 12, 15, 17, 21} ______________

   (b) {0} ______________

   (c) { } ______________

   (d) {1, 2, 3, ..., 13} ______________

   (e) {15, 16, 17, ..., 23} ______________

2. Classify each of the following sets as being finite, infinite, or null.

   (a) {5, 6, 7, ..., 999} ______________

   (b) {5, 6, 7, ...} ______________

   (c) set of whole numbers less than zero ______________

   (d) set of cities in the world ______________

   (e) set of odd natural numbers ______________

C. Subsets

> One set is a SUBSET of another set if all its elements are contained in the larger set.

EXAMPLES:

1. The set of all cats is a subset of the set of all animals since every cat is also an animal.
2. $A = \{3, 4, 5\}$ is a subset of $B = \{1, 2, 3, 4, 5, 6\}$ since every element in set A also belongs to set B.

> The null set is a subset of every set. Also, every set is a subset of itself.

Suppose we wished to write all the subsets of a given set. For example, consider the set $M = \{a, b, c\}$ and all its possible subsets.

1. In forming subsets of M, we could choose any one of the three elements of M. This would give us the subsets:
   $\{a\}, \quad \{b\}, \quad \{c\}$
2. We could also form subsets by choosing any two of the three elements of M. This would give us the subsets:
   $\{a, b\}, \quad \{a, c\}, \quad \{b, c\}$ } Remember that the order of listing elements is not important.
3. We could choose all three members of M and obtain the subset, $\{a, b, c\}$.
4. We must not forget the null set, $\{ \}$, which is also a subset of M.

Thus, we can form eight possible subsets of $M = \{a, b, c\}$. These eight subsets are:

$$\{a\}, \ \{b\}, \ \{c\}, \ \{a, b\}, \ \{a, c\}, \ \{b, c\}, \ \{a, b, c\}, \ \{ \}$$

All these subsets except $\{a, b, c\}$ are called PROPER SUBSETS of M. A proper subset cannot contain all the elements of the original set. To indicate that one set is a proper subset of another, we use the symbol "$\subset$". In this case, we can write:

$$\{a\} \subset \{a, b, c\}$$
$$\{a, b\} \subset \{a, b, c\} \quad \text{and so on.}$$

$\{a, b, c\}$ is not a proper subset of set M because it contains all the elements of set M.

Self-correcting Exercise #4

Answers to this exercise may be found on page 34 of this lesson.

1. $A = \{1, 2, 3, \ldots, 10\}$. Decide whether or not each of the following sets is a subset of set A. (Write "yes" if it is and "no" if it isn't).

(a) $\{1, 2, 3\}$ _____ (f) $\{4, 5, 9, 10, 11\}$ _____

(b) $\{6, 8, 10, 12\}$ _____ (g) $\{1, 2, 3, \ldots, 10\}$ _____

(c) $\{4, 5, 6\}$ _____ (h) $\{10\}$ _____

(d) $\{0, 1, 2, 3, 4\}$ _____ (i) $\{\ \}$ _____

(e) $\{2, 3, 4, 5, 6, 7, 8\}$ _____ (j) $\{0\}$ _____

2. $R = \{1, 2\}$

(a) List the two subsets of R that contain 1 element.

_____, _____

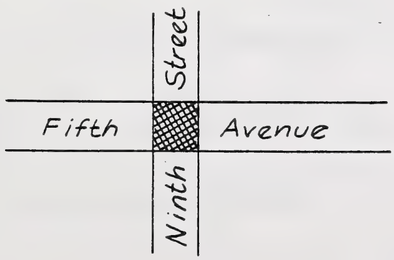
(b) List the one subset of R that contains 2 elements. _____

(c) List the one subset of R that contains no elements. _____

(d) Which of these subsets is <u>not</u> a proper subset of R? _____

D. Intersection of Sets

We use the word "intersection" when we talk about the place where two roads meet and cross.



The cross-hatched region in the diagram represents the intersection of Fifth Avenue and Ninth Street.

We can also use the word "intersection" when we are talking about sets of objects.

> The INTERSECTION of two sets is a set containing all those elements which the two sets have in common.

We use the symbol "$\cap$" to represent the intersection of two sets.

EXAMPLE: If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5, 6\}$, find $A \cap B$.
Solution
THIS IS READ "A INTERSECTION B".

What elements do set A and set B have in common?
Note that the elements 2 and 4 belong to both sets.
Therefore, they must belong to the intersection set, $A \cap B$.

$$A \cap B = \{2, 4\}.$$

THESE ELEMENTS ARE COMMON TO BOTH SETS.

If two sets have no elements in common, their intersection set is $\{\ \}$, the null set. For example, the sets $M = \{a, b, c\}$ and $N = \{d\}$ have no elements in common. Therefore, $M \cap N = \{\ \}$.

Self-correcting Exercise #5

Answers to the exercise may be found on page 34 of this lesson.

1. For each exercise decide what elements A and B have in common. Then tabulate $A \cap B$. (Remember to put curly braces around each intersection set.)

(a) $A = \{4, 5, 6, 10, 12\}$ $B = \{4, 6, 10, 14\}$
What elements do A and B have in common? ______________

$A \cap B =$ ______________________

(b) $A = \{a, b, c, d, e, f, g\}$ $B = \{c, d, f, \}$
What elements do A and B have in common? ______________

$A \cap B =$ ______________________

(c) $A = \{1, 2, 3\}$ $B = \{1, 2, 3\}$
What elements do A and B have in common? ______________

$A \cap B =$ ______________________

(d) $A = \{\text{blue}, \text{green}\}$ $B = \{\text{yellow}\}$
What elements do A and B have in common? ______________

$A \cap B =$ ______________________

(e) $A = \{1, 2, 3, \ldots, 10\}$ $B = \{6, 7, 8, \ldots, 15\}$
What elements do A and B have in common? ______________

$A \cap B =$ ______________________

E. Union of Sets

We use the word "union" when we talk about two or more groups combining to form a single group. The new group that is formed from the union contains all the members that belong to any of the original groups.

In mathematics, we can talk about the "union" of sets.

The UNION of two sets is a set containing any element that belongs to either set.

We use the symbol "$\cup$" to represent the union of two sets. It may help you to remember this symbol if you associate it with "u", the first letter in "union".

When tabulating a union set, you must be careful not to repeat elements that belong to both sets. These elements should be listed only once.

THIS IS READ "R UNION S."

EXAMPLE: If $R = \{2, 4, 6, 8\}$ and $S = \{4, 8, 12, 16, 20\}$, find $R \cup S$.

Solution

What elements belong to set R, set S, or both? Note that the elements 2, 4, 6 and 8 belong to set R so they must also belong to the union set. Now, look at the elements in set S. Since the elements 4 and 8 have already been placed in the union set, we need only add the remaining elements 12, 16, and 20.

$$R \cup S = \{2, 4, 6, 8, 12, 16, 20\}$$

Self-correcting Exercise #6

Answers to this exercise may be found on page 34 of this lesson.

1. For each exercise, state the elements that set A contributes to the union set, state the new elements that set B contributes, and then tabulate $A \cup B$.

   (a) $A = \{4, 5, 6, 10, 12\}$, $B = \{4, 6, 10, 14\}$

   Elements which set A contributes ____________________

   New elements which set B contributes ____________________

   $A \cup B$ = ______________________________ (REMEMBER TO PUT BRACES AROUND THE SET.)

(b) $A = \{c, d, f\}$, $B = \{a, b, c, d, e, f, g\}$

Elements which set A contributes ________________

New elements which set B contributes ________________

$A \cup B =$ ____________________________

(c) $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$

Elements which set A contributes ________________

New elements which set B contributes ________________

$A \cup B =$ ____________________________

(d) $A = \{\text{blue, green}\}$, $B = \{\text{yellow}\}$

Elements which set A contributes ________________

New elements which set B contributes ________________

$A \cup B =$ ____________________________

(e) $A = \{1, 2, 3, \ldots, 10\}$, $B = \{6, 7, 8, \ldots, 15\}$

Elements which set A contributes ________________

New elements which set B contributes ________________

$A \cup B =$ ____________________________

F. Cartesian Product of Sets

The CARTESIAN PRODUCT of set A and set B is a set containing all those ordered pairs that are formed by matching each member of set A in turn with each member of set B.

We use the symbol $A \times B$, which is read, "A cross B", to represent the Cartesian product of set A and set B.

The members of set A are the first components of the ordered pairs and the members of B are the second components.

An ORDERED PAIR consists of two objects that occur in a special order. The object which appears first is called the FIRST COMPONENT of the ordered pair. The object which appears second is called the SECOND COMPONENT. Components of ordered pairs are always written inside a set of parentheses. A comma is placed between the two components. For example, (Paris, France) is an ordered pair. It is referred to as "the ordered pair Paris, France." The first component of this ordered pair is "Paris" and the second component is "France".

Write the ordered pair France, Paris. ______________________

In this case, the first component is "__________" and the second component is "__________". This is an entirely different ordered pair than (Paris, France).

Write the ordered pair whose first component is 3 and second component is 5. ____________________

Write the ordered pair whose first component is 5 and second component is 3. ____________________

EXAMPLE: At an ice-cream parlour, one can buy a sundae which consists of a scoop of ice-cream with some topping on it. Suppose that there are three kinds of ice-cream available which we will consider as members of the set I.

I = {vanilla, maple walnut, strawberry}

There are two types of topping available which we will consider to be members of the set T.

T = {butterscotch, chocolate}

We can form the Cartesian set $I \times T$ to help us determine the different kinds of sundaes that can be made. A TREE DIAGRAM can be used to construct all the ordered pairs that will appear in $I \times T$. Note that each member of set I is paired in turn with each member of set T. Fill in the blanks in the tree diagram below.

Elements of Set I	Elements of Set T	Elements of Set $I \times T$
vanilla	butterscotch	→ (vanilla, butterscotch)
	chocolate	→ (vanilla, chocolate)
maple walnut	butterscotch	→ (maple walnut, ________)
	chocolate	→ (________, chocolate)
strawberry	butterscotch	→ (strawberry, butterscotch)
	chocolate	→ (strawberry, ________)

The Cartesian set $I \times T$ could be tabulated as follows:

Curly braces are placed around the entire set.
Parentheses are placed around each ordered pair.

$I \times T$ = {(vanilla, butterscotch), (vanilla, chocolate), (maple walnut, butterscotch), (maple walnut, chocolate), (strawberry, butterscotch), (strawberry, chocolate)}

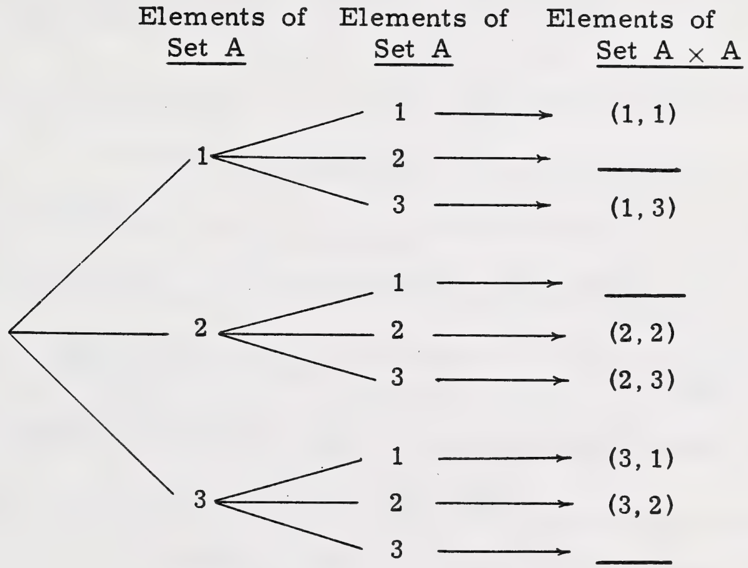
Note that this set contains SIX elements. It has a different appearance than the sets we have dealt with so far since the elements are all ordered pairs rather than single objects.

For each ordered pair in the Cartesian set $I \times T$, the first component tells us the type of ice-cream used and the second component tells us the type of topping used. For example, the ordered pair (vanilla, butterscotch) represents a sundae with vanilla ice-cream and butterscotch topping.

What does the ordered pair (strawberry, chocolate) represent?

____

We can also form the Cartesian product of a set with itself. For example, if we are given the set $A = \{1, 2, 3\}$, we can form the Cartesian product $A \times A$ by matching each member of set A in turn with all the members of set A. Both the first and second components of the ordered pairs will be elements of set A. A tree diagram can be used to determine all the ordered pairs that will appear in the Cartesian set $A \times A$. Fill in the blanks in the tree diagram below.



Therefore, $A \times A = \{(1, 1), (\_\_\_), (1, 3), (\_\_\_), (2, 2), (2, 3), (3, 1), (3, 2), (\_\_\_)\}$.

How many ordered pairs does set $A \times A$ contain? ________

Self-correcting Exercise #7

Answers to this exercise may be found on page 35 of this lesson. Check your answers very carefully to make sure you have used curly braces, parentheses, and commas correctly.

1. $A = \{1, 2, 3, 4\}$ and $B = \{7, 8, 9\}$.

   (a) Tabulate $A \times B$.

   $A \times B =$ ______________________________

   (b) Tabulate $B \times A$.

   $B \times A =$ ______________________________

2. $S = \{H, T\}$. Tabulate $S \times S$.

   $S \times S =$ ______________________________

EXERCISE - Sets

1. Fill in the blanks. Use the words from this list.

subset	commas	cross	braces
element	proper	well-defined	cardinal
intersection	null	tabulate	Cartesian
union	infinite	natural	component
ordered	finite	whole	set

(a) "$\cap$" is the symbol for ______________________.

(b) The ______________ set is a subset of every set.

(c) $\{0, 1, 2, \ldots\}$ is the set of ______________ numbers.

(d) When we tabulate sets, we must put ______________ between the elements.

(e) $A = \{0, 2\}$ is a ______________ of $B = \{0, 1, 2, 3\}$.

(f) Any object which belongs to a given set is called an ______________ of that set.

(g) The number of elements in a set is called its ________________ number.

(h) $R \times S$ is read, "R __________ S" and represents the ____________ product of R and S. Each element of $R \times S$ is an ______________ pair.

(i) A set with a limited number of members is called a __________________ set.

(j) When we __________________ a set, we list its elements within braces.

(k) Set A is not a __________ subset of set B if it contains exactly the same elements as set B.

(l) A set is __________________ if it is possible to tell without doubt what elements belong to the set.

(m) The ____________________ of A = {1, 2} and B = {1, 5, 7} is {1, 2, 5, 7}.

(n) In the ordered pair (1, 5), the number 1 is called the first ______________ of the ordered pair.

2. If A = {0, 10, 20}, B = {1, 4, 9, 16}, and C = {3, 12}, tabulate the following Cartesian sets.

(a) $A \times B$ = { (0, 1), ____________________

____

(b) $B \times A$ = { ____________________

____

(c) $A \times C$ = { ____________________

(d) $C \times C$ = { ____________________

(e) $C \times B$ = { ____________________

3. Tabulate each of the following sets. For very large or infinite sets, use 3 dots to indicate omitted members.) Then, decide if the set is finite, infinite, or null.

(a) A, the set of natural numbers between 5 and 75.

Tabulation A = {6, 7, 8, ..., 74}

Classification finite

(b) B, the set of days of the week whose names begin with the letter "S".

Tabulation ____________________

Classification ____________________

(c) C, the set of natural numbers greater than 10.
(Hint: 10 does not belong to this set)

Tabulation ______________________________

Classification ______________________________

(d) D, the set of whole numbers less than 93.
(Hint: 0 is the first element and 92 is the last.)

Tabulation ______________________________

Classification ______________________________

(e) E, the set of people in your family that are over 10 ft. tall.

Tabulation ______________________________

Classification ______________________________

(f) F, the set of vowels in the alphabet.
(Do not include "y".)

Tabulation ______________________________

Classification ______________________________

(g) G, the set of even natural numbers between 100 and 200.
(Hint: 100 and 200 are not included.)

Tabulation ______________________________

Classification ______________________________

(h) H, the set of months in the year having exactly 30 days.

Tabulation ______________________________

Classification ______________________________

(i) I, the set of months in the year having 32 days.

Tabulation ____________________

Classification ____________________

(j) J, the set of natural numbers that are divisible by 7.

Tabulation ____________________

Classification ____________________

(k) K, the set of letters in the word "prairie".

Tabulation ____________________

Classification ____________________

4. For each exercise, decide whether or not the objects listed belong to set A. Write "yes" in the blank if the object belongs to set A and "no" if it doesn't.

(a) A is the set of Canadian coins.

nickel	yes	sixpence	_____
half-dollar	_____	quarter	_____
$10 bill	_____	franc	_____
dime	_____	pound	_____

(b) $A = \{4, 6, 8, \ldots, 24\}$

2	no	9	_____
4	_____	10	_____
5	_____	16	_____
7	_____	25	_____

(c) A is the set of cities in the world.

New York	______	Alberta	______
Italy	______	Florida	______
Asia	______	London	______
Ottawa	______	Paris	______

5. Give the cardinal number of each of these sets.

(a) $\{2, 4, 6\}$ ______

(b) $\{2, 4, 6, \ldots, 14\}$ ______

(c) $\emptyset$ ______

(d) $\{a\}$ ______

(e) $\{1, 2, 3, \ldots, 100\}$ ______

6. Using the elements of $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, tabulate each of the following subsets of U.

(a) The set of even numbers in U.

$\{2, 4, 6, 8, 10\}$

(b) The set of odd numbers in U.

______

(c) The set of numbers in U that are less than 6.

______

(d) The set of numbers in U that are between 3 and 9.

______

(e) The set of numbers in U that are multiples of 3.

______

(f) The set of numbers in U that are greater than or equal to 4.

______

7. For each pair of sets, write the intersection set, $A \cap B$, and the union set, $A \cup B$.

(a) $A = \{m, n, o\}$, $B = \{m, o, p, q, r\}$

$A \cap B =$ $\{m, o\}$ ← THESE ELEMENTS BELONG TO BOTH A AND B.

$A \cup B =$ $\{m, n, o, p, q, r\}$ ← THESE ELEMENTS BELONG TO EITHER A OR B.

(b) $A = \{2, 3, 4, 5, 6\}$, $B = \{2, 3, 6, 8, 10\}$

$A \cap B =$ __________

$A \cup B =$ __________

(c) $A = \{\text{red}\}$, $B = \{\text{blue}\}$

$A \cap B =$ __________

$A \cup B =$ __________

(d) $A = \{r, s, t, v\}$, $B = \{o, p, q, r, s, t\}$

$A \cap B =$ __________

$A \cup B =$ __________

(e) $A = \{t, o, r\}$, $B = \{o, r, t\}$

$A \cap B =$ __________

$A \cup B =$ __________

8. Decide whether or not each of the following sets is a subset of $N = \{1, 2, 3, \ldots\}$. (Write "yes" if it is and "no" if it isn't).

(a) $\{4, 5, 6, \ldots, 12\}$ yes

(b) $\{1\}$ _____

(c) $\{0\}$ _____

(d) $\{\ \}$ _____

(e) $\{3, 6, 9\}$ _____

(f) $\{1, 2, 3, \ldots\}$ _____

(g) $\{0, 1, 2, 3, \ldots\}$ _____

(h) $\{-1, -2, -3\}$ _____

(i) $\{\frac{1}{2}, \frac{1}{4}\}$ _____

(j) $\{10, 20, 30, \ldots\}$ _____

9. If you have the following set of sweaters to choose from:

$$A = \{\text{white, black, yellow}\}$$

and the following set of slacks:

$$B = \{\text{brown, navy, black}\},$$

tabulate the Cartesian set $A \times B$ which represents the different outfits you can wear.

$A \times B$ = {(white, brown), ____

____ }

The first component of each ordered pair belongs to set _____ and represents the colour of __________ you are wearing.

The second component of each ordered pair belongs to set _____ and represents the colour of __________ you are wearing.

The ordered pair (white, brown) represents an outfit composed of a white ____________ and brown ____________.

10. If $A = \{7, 11, 14\}$, list the eight possible subsets of A.

{ }, {7}, _____, _____, _____, _____, _____, _____

Which subset is <u>not</u> a proper subset? ____________________.

Topic Two: The Set of Natural Numbers and the Set of Whole Numbers

The set of natural numbers contains the counting numbers 1, 2, 3, 4, 5, 6, 7, and so on. We use the following notation to represent the infinite set of natural numbers.

$$N = \{1, 2, 3, \dots\}$$

What capital letter has been used to represent this set? ____________

What do the three dots mean? ______________________________

____

Does 5 belong to set N? _____ Does -2? _____ Does 2160? _____

Does 0? _____ Does 1.7? _____ Does $3\frac{1}{2}$? _____ Does 100? _____

The number zero was not invented until many centuries after the counting numbers. Such a number was needed to convey the idea of "nothingness". Zero is the number which tells us "not any".

When we put the number zero with the natural numbers, we obtain the set of whole numbers. We use the following notation to represent the infinite set of whole numbers.

$$W = \{0, 1, 2, 3, \dots\}$$

What capital letter has been used to represent this set? ____________

Does 5 belong to set W? _____ Does -2? _____ Does 2160? _____

Does 0? _____ Does 1.7? _____ Does $3\frac{1}{2}$? _____ Does 100? _____

A. The Whole Number Line

The set of whole numbers may be associated with points on a straight line. If a line is divided into equal segments, we can associate the ends of these segments with whole numbers.

The above line has been divided into equal segments. Going from left to right, the endpoints of the first segment have been labeled 0 to 1. Note how points to the right of 0 and 1 have been labeled with successive natural numbers. You can see how each whole number can be associated with a point on the line. An arrowhead is drawn on the right of the line to indicate that it extends indefinitely to the right.

In drawing a number line, we can freely choose the unit of measure to be used in scaling the line. But, regardless of the unit of measure used, the scale must be uniform. Thus, the distance between the points labeled 0 and 1 must be the same as the distance between the points labeled 1 and 2, and so on. The number that is paired with a point on the number line is called the COORDINATE (ko-or´-din-it) of that point.

In the space provided below:

1. Draw a line about 14 cm long.
2. Mark off segments of 1 cm each.
3. Associate a whole number with each division on the line.
4. Draw an arrowhead at the right end of the line.

WHOLE NUMBER LINE

B. Ordering of the Whole Numbers

In dealing with any two whole numbers "a" and "b", we find that exactly one of the following three possibilities must be true.

1. $a = b$ (The numbers are equal.)
2. $a > b$ ("a" is greater than "b".)
3. $a < b$ ("a" is less than "b".)

This is called the TRICHOTOMY PROPERTY ("trichotomy" means "divided into three parts".)

We can tell how two whole numbers are ordered by their positions on the number line. Numbers named further to the right are larger. For example, 6 is greater than 3 since 6 lies to the right of 3 on the number line. We can write:

$6 > 3$ ⟵ This is read, "6 is greater than 3".

We know that 2 is less than 7 since 2 lies to the left of 7 on the number line. We can write:

$2 < 7$ ⟵ This is read, "2 is less than 7".

Number sentences which involve the symbols ">" and "<" are called <u>inequalities</u>.

> In using the symbols $>$ and $<$, remember that the point of the arrowhead is always directed toward the smaller number.
>
> $4 < 6$ — Arrowhead points toward the smaller number 4.
>
> $6 > 4$

<u>Self-correcting Exercise #8</u>

Answers to this exercise may be found on page 35 of this lesson.

1. Insert the symbol $>$ or $<$ between the following pairs of whole numbers.

(a)	11 ______	15	(b)	0 ______	1	(c)	232 ______	231
(d)	110 ______	120	(e)	13 ______	130	(f)	12 ______	0
(g)	1001 ______	1000	(h)	101 ______	110	(i)	900 ______	90
(j)	2 ______	1	(k)	12 ______	21	(l)	10 ______	100

2. Write each inequality in words. (Write the numerals as well as the symbols in words.)

(a) $23 < 32$

____

(b) $1000 > 100$

____

(c) $40 < 98$

____

C. Set-Builder Notation

Sometimes instead of tabulating a set (listing its elements within braces), we specify it by using SET-BUILDER NOTATION. This notation allows us to state a rule for membership in the set.

For example, suppose set C is the set of all whole numbers less than 5. This set could be tabulated as $C = \{0, 1, 2, 3, 4\}$, or it could be expressed in set-builder notation as follows:

$$C = \{x \mid x < 5, x \in W\}$$

This is read, "C is the set of all x such that x is less than 5 and x is a whole number." Note that:

1. The letter x was used to represent all the elements of set C. The statement "x is less than 5" is true whenever x is replaced by an element of set C.

2. The word "all" is used in reading this notation even though it is not actually written.

3. The vertical line "|" which is called a solidus is a handy abbreviation for the words "such that".

4. The symbol $\{x \mid \qquad\qquad \}$ is called the SET BUILDER. In the blank space we mention a property or characteristic common to all members of the set.

5. The symbol "ϵ" is the Greek letter "epsilon" and is used to mean "is an element of". $x \in W$ means that x is an element of set W; in other words, x is a whole number.

Now let us specify the following sets using set-builder notation.

1. The set of all whole numbers greater than or equal to 8.

   Set-builder notation: $\{x \mid x \geqq 8, x \in W\}$.

   SYMBOL FOR "IS GREATER THAN OR EQUAL TO"

   This is read, "the set of all x such that x is greater than or equal to 8 and x is a whole number."

2. The set of all whole numbers less than or equal to 9.

   Set-builder notation: ______________________

   This is read, " ______________________

   ______________________ "

In cases where x lies between two boundaries we can show this by using two inequality symbols. Note how the following statements can be written symbolically.

Statement	Symbolic Representation
1. x is between 9 and 15	$9 < x < 15$ This can be read "x is greater than 9 and less than 15."
2. x is between 9 and 15 inclusive	$9 \leqq x \leqq 15$ This can be read "x is greater than or equal to 9 and less than or equal to 15."
3. x is greater than or equal to 9 and less than 15.	$9 \leqq x < 15$
4. x is greater than 9 and less than or equal to 15.	$9 < x \leqq 15$

> **IMPORTANT!**
>
> In the above examples, the values of x fall between two boundaries. You must decide which number is smaller and write it to the left of x. The larger number is written to the right of x. Both arrowheads point to the left.

Self-correcting Exercise #9

Answers to this exercise may be found on page 35 of this lesson.

1. Write the following statements in symbols.

(a) x is greater than or equal to 10. $x \geqq 10$

(b) x is between 4 and 15 inclusive. ______________

(c) x is greater than or equal to 3 and less than 13. ______________

(d) x is less than 19. ______________

(e) x is between 40 and 50. ______________

(f) x is larger than 33. ______________

(g) x is greater than 1 and less than 5. ______________

(h) x is greater than 2 and less than or equal to 8. ______________

(i) x is less than or equal to 17. ______________

2. Specify the following sets using set-builder notation. Then tabulate each set.

(a) The set of all whole numbers between 13 and 1,000.

Set-builder notation: $\{x \mid 13 < x < 1000,\ x \in W\}$

Tabulation: $\{14, 15, 16, \ldots, 999\}$

(b) The set of all whole numbers greater than or equal to 50.

Set-builder notation: ______________

Tabulation: ______________

(c) The set of all whole numbers greater than or equal to 5 and less than 10.

Set-builder notation: ______________

Tabulation: ______________

(d) The set of all whole numbers between 100 and 200 inclusive.

Set-builder notation: ______________

Tabulation: ______________

(e) The set of all whole numbers less than or equal to 5.

Set-builder notation: ______________

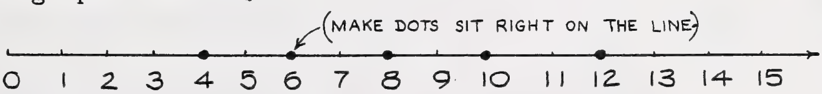
Tabulation: ______________

D. <u>Graphing Subsets of W</u>

The graph of a given subset of the set of whole numbers is the set of points on the number line whose coordinates are members of the given set of numbers. To graph a subset of W, we make heavy dots on the number line at the points corresponding to the elements of the subset. The process of making dots on the number line at points corresponding to members of a given set is called GRAPHING THE SET.

Study the following examples in order to learn more about graphing subsets of W.

EXAMPLE 1: Graph the set $C = \{4, 6, 8, 10, 12\}$.
<u>Solution</u>. To graph the set C, we make heavy dots at the points on the number line corresponding to the numbers 4, 6, 8, 10, and 12. We say that the set of five points indicated by these five dots is the graph of set C.

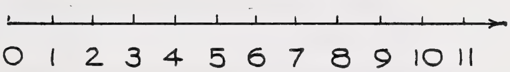


GRAPH OF $C = \{4, 6, 8, 10, 12\}$

EXAMPLE 2: Draw the graph of the set $M = \{x \mid 2 < x \leqq 8, x \epsilon W\}$.
<u>Solution</u>. Since set M contains all whole numbers which are greater than 2, and less than or equal to 8, it will contain the elements 3, 4, 5, 6, 7, and 8. Tabulate set M.

M = ______________________

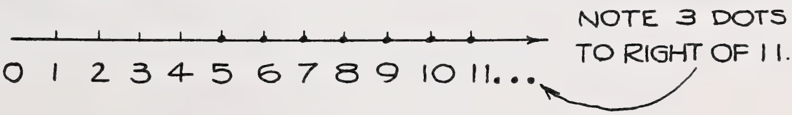
We can graph set M by placing heavy dots at the points corresponding to the numbers 3, 4, 5, 6, 7, and 8. Graph set M on the number line below.



EXAMPLE 3: Draw the graph of the set $R = \{x \mid x > 4, x \epsilon W\}$
<u>Solution</u>. Note that set R could be tabulated as follows:

$$R = \{5, 6, 7, \ldots\}$$

Since R is an infinite set, it is impossible to graph it in its entirety. As a compromise, we graph the first few elements of the set and then place 3 dots to the right of the last number listed to indicate that the graph goes on indefinitely to the right, following the same pattern.



Self-correcting Exercise #10

Answers to this exercise may be found on page 36 of this lesson.

For each of the following questions, draw a whole number line and graph the given set on the line.

1. $A = \{3, 5, 11\}$

   Graph:

2. $B = \{5, 6, 7, \ldots\}$

   Graph:

3. $C = \{x \mid x \leqq 9,\ x \in W\}$

   Tabulate set C. ____________________

   Graph:

4. $D = \{x \mid 3 < x \leqq 11,\ x \in W\}$

   Tabulate set D. ____________________

   Graph:

5. $E = \{x \mid x \geqq 2,\ x \in W\}$

   Tabulate set E. ____________________

   Graph:

EXERCISE-Natural Numbers and Whole Numbers

1. Insert the symbol > or < between the following pairs of numbers.

   (a) 100 000 _____ 101 000 (b) 17 _____ 16

   (c) 203 _____ 230 (d) 5 _____ 0

2. In each question, compare the two numbers and decide how they are related to each other. Then, put a check mark in one of the last three columns.

a	b	a = b	a < b	a > b
(a) 13	9			✓
(b) 5	6			
(c) 20 + 1	22 - 1			
(d) 30 ÷ 5	45 ÷ 9			
(e) (8 + 4) ÷ 3	(4 + 4) ÷ 2			
(f) (3 × 5) + 1	34 ÷ 2			
(g) 5 - 2	2 × 2			

3. In each question, fill in the first blank with the symbol "<" or ">" and the second blank with the word "right" or "left".

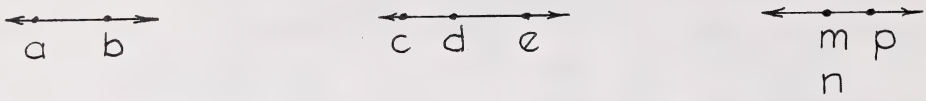
   (a) 5 _>_ 2 since 5 lies to the _right_ of 2 on the number line.

   (b) 6 ___ 9 since 6 lies to the __________ of 9 on the number line.

   (c) 4 + 4 ___ 3 + 2 since 4 + 4 lies to the __________ of 3 + 2 on the number line.

   (d) 9 - 7 ___ 2 + 1 since 9 - 7 lies to the __________ of 2 + 1 on the number line.

   (e) 4 ___ 27 ÷ 9 since 4 lies to the __________ of 27 ÷ 9 on the number line.

4. 

With reference to the above graphs, are the following statements true or false?

True or False?

(a) $b > a$ __________

(b) $d < e$ __________

(c) $c > e$ __________

(d) $n > p$ __________

(e) $e > d$ __________

(f) $m = n$ __________

(g) $m < p$ __________

(h) $d < c$ __________

5. Write the following statements in symbols.

(a) x is between 5 and 9 inclusive. $5 \leqq x \leqq 9$

(b) x is less than 23. __________

(c) x is less than 12 and greater than 4. __________

(d) x is greater than or equal to 14. __________

(e) x is greater than or equal to 3 and less than 21. __________

(f) x is between 13 and 7. __________

(g) x is greater than 1 and less than or equal to 15. __________

(h) x is less than or equal to 50 and greater than or equal to 25. __________

(i) x is greater than 100. __________

6. Write the following symbolic statements in words.

(a) $4 < x \leqq 9$

x is greater than 4 and less than or equal to 9.

(b) $x < 33$

(c) $2 \leqq x < 50$

(d) $10 < x < 25$

(e) $x \geqq 14$

(f) $90 \leqq x \leqq 100$

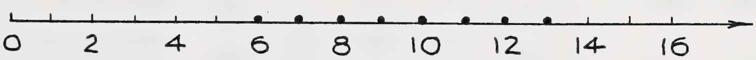
7. Specify each set by using set-builder notation. Then tabulate the set and draw its graph on the whole number line.

(a) The set of all whole numbers between 6 and 13 inclusive.

Set-builder notation: $\{x \mid 6 \leqq x \leqq 13, x \in W\}$

Tabulation: $\{6, 7, 8, \ldots, 13\}$

Graph:



(b) The set of all whole numbers greater than 5.

Set-builder notation:

Tabulation:

Graph:

(c) The set of all whole numbers greater than or equal to 3 and less than 13.

Set-builder notation: ______________________________

Tabulation: ______________________________

Graph:

(d) The set of all whole numbers less than 6 and greater than 1.

Set-builder notation: ______________________________

Tabulation: ______________________________

Graph:

(e) The set of all whole numbers less than or equal to 7.

Set-builder notation: ______________________________

Tabulation: ______________________________

Graph:

Key to Self-correcting Exercises in Lesson 1

Exercise #1, page 2

1. (a) {5, 30, 95}

   (b) {Sun., Mon., Tues., Wed., Thurs., Fri., Sat.}

   (c) {a, t, e, n, i, o} ← EACH LETTER IS LISTED ONLY ONCE.

   The elements may be listed in any order.

   (d) {1, 0, 3, 5} ← EACH NUMERAL IS LISTED ONLY ONCE.

2. (a) C is the set whose elements are 8, 3, and 2.

   (b) R is the set whose elements are "a" and "b".

Exercise #2, page 3

1. (a) {1, 2, 3, . . . , 129} FIRST NATURAL NUMBER; LAST NUMBER IS LESS THAN 130.

   (b) {0, 1, 2, . . . , 129} FIRST WHOLE NUMBER

   (c) {26, 27, 28, . . .} GREATER THAN 25

   (d) {18, 24, 30, . . . , 144} (NOTE THAT 12 AND 150 ARE NOT INCLUDED)

   (e) {5, 6, 7, . . . , 200} (NOTE THAT 5 AND 200 ARE INCLUDED)

2. M = {5, 10, 15, . . . , 95}. Set M contains all multiples of 5 between 5 and 95 inclusive.

(a)	5	yes	(b)	6	no	(c)	12	no	(d)	16	no
(e)	20	yes	(f)	29	no	(g)	30	yes	(h)	92	no
(i)	65	yes	(j)	96	no	(k)	100	no	(l)	0	no

Exercise #3, page 5

1.	(a)	6	(b)	1 (1 ELEMENT)	(c)	0 (NO ELEMENTS)	(d)	13	(e)	9
2.	(a)	finite	(b)	infinite	(c)	null	(d)	finite	(e)	infinite

Exercise #4, page 7

1. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

   (a) yes
   (b) no (12 does not belong to set A.)
   (c) yes
   (d) no (0 does not belong to set A.)
   (e) yes
   (f) no (11 does not belong to set A.)
   (g) yes (Every set is a subset of itself.)
   (h) yes
   (i) yes (The null set is a subset of every set.)
   (j) no (0 does not belong to set A.)

2. (a) $\{1\}$, $\{2\}$
   (b) $\{1, 2\}$ ← (This is the same as $\{2, 1\}$ since elements can be listed in any order.)
   (c) $\{\ \}$ or $\emptyset$
   (d) $\{1, 2\}$ Not a proper subset because it contains all the elements of set R.

Exercise #5, page 8

1. (a) A and B have the elements 4, 6, and 10 in common.
   $A \cap B = \{4, 6, 10\}$
   (b) A and B have the elements c, d, and f in common.
   $A \cap B = \{c, d, f\}$
   (c) A and B have the elements 1, 2, and 3 in common.
   $A \cap B = \{1, 2, 3\}$
   (d) A and B have no elements in common.
   $A \cap B = \{\ \}$
   (e) $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.
   A and B have the elements 6, 7, 8, 9, 10 in common.
   $A \cap B = \{6, 7, 8, 9, 10\}$

Exercise #6, page 9

1. (a) Set A contributes 4, 5, 6, 10, and 12. Set B contributes the new element, 14.
   $A \cup B = \{4, 5, 6, 10, 12, 14\}$
   (b) Set A contributes the elements c, d, and f. Set B contributes the new elements a, b, e, and g.
   $A \cup B = \{c, d, f, a, b, e, g\}$
   (c) Set A contributes the elements 1, 2, and 3. Set B contributes no new elements.
   $A \cup B = \{1, 2, 3\}$
   (d) Set A contributes the elements blue and green. Set B contributes the new element, yellow.
   $A \cup B = \{\text{blue, green, yellow}\}$

(e) Set A contributes the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.
Set B contributes the new elements 11, 12, 13, 14, 15.
$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$
(or) $= \{1, 2, 3, \ldots, 15\}$

Exercise #7, page 13

1. (a) $A \times B = \{(1,7), (1,8), (1,9), (2,7), (2,8), (2,9), (3,7), (3,8), (3,9), (4,7), (4,8), (4,9)\}$

Note that all the first components of the ordered pairs belong to set A and all the second components belong to set B.

(b) $B \times A = \{(7,1), (7,2), (7,3), (7,4), (8,1), (8,2), (8,3), (8,4), (9,1), (9,2), (9,3), (9,4)\}$

Note that all the first components of the ordered pairs belong to set B and all the second components belong to set A.

(c) $S \times S = \{(H,H), (H,T), (T,H), (T,T)\}$

Exercise #8, page 23

1. (a) $11 < 15$ (b) $0 < 1$ (c) $232 > 231$

(d) $110 < 120$ (e) $13 < 130$ (f) $12 > 0$

(g) $1001 > 1000$ (h) $101 < 110$ (i) $900 > 90$

(j) $2 > 1$ (k) $12 < 21$ (l) $10 < 100$

2. (a) Twenty-three is less than thirty-two.

(b) One thousand is greater than one hundred.

(c) Forty is less than ninety-eight.

Exercise #9, page 25

1. (a) $x \geqq 10$ (b) $4 \leqq x \leqq 15$ (c) $3 \leqq x < 13$

(d) $x < 19$ (e) $40 < x < 50$ (f) $x > 33$

(g) $1 < x < 5$ (h) $2 < x \leqq 8$ (i) $x \leqq 17$

Exercise #9, cont'd

2. (a) $\{x \mid 13 < x < 1000,\ x \in W\}$; $\{14, 15, 16, \ldots, 999\}$

(b) $\{x \mid x \geqq 50,\ x \in W\}$; $\{50, 51, 52, \ldots\}$

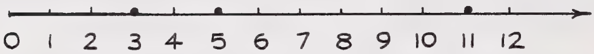
(c) $\{x \mid 5 \leqq x < 10,\ x \in W\}$ $\{5, 6, 7, 8, 9\}$

(d) $\{x \mid 100 \leqq x \leqq 200,\ x \in W\}$ $\{100, 101, 102, \ldots, 200\}$

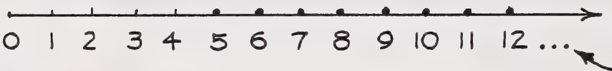
(e) $\{x \mid x \leqq 5,\ x \in W\}$ $\{0, 1, 2, 3, 4, 5\}$

Exercise #10, page 28

1.

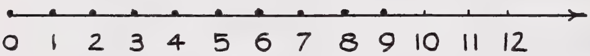


2.

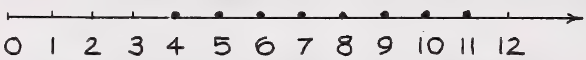


THE 3 DOTS INDICATE THAT THE GRAPH GOES ON INDEFINITELY TO THE RIGHT.

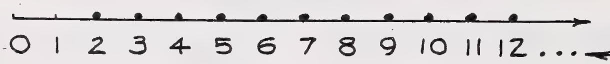
3. C = $\{0, 1, 2, \ldots, 9\}$



4. D = $\{4, 5, 6, 7, 8, 9, 10, 11\}$



5. E = $\{2, 3, 4, \ldots\}$



NOTE 3 DOTS.

End of Lesson 1

Lesson 2

Properties of Whole Numbers

Basic Algebra and Geometry

PROPERTIES OF WHOLE NUMBERS

You will likely find that there is alot of new material for you to learn in this lesson. Take your time with the lesson and try to memorize the new terms as you go along.

Topic One: Properties of Whole Numbers Under Addition and Multiplication

A. Operations of Addition and Multiplication

Addition and multiplication are two of the fundamental operations of arithmetic. They are called binary operations because they are performed on two numbers at a time.

Addition is the operation that assigns to two numbers a third number called the SUM. The two numbers that are added are called ADDENDS. The operation of addition is indicated by placing a plus sign (+) between the pair of numbers to be added.

EXAMPLE:

$$5 + 7 = 12$$

ADDENDS SUM

Fill in the blanks below.

1. In the number sentence 1 + 9 = 10, the addends are ________ and ________, and the sum is ________. The symbol ________ is used to indicate that the operation is addition.

2. What is the sum of 12 and 13? __________

3. Give four different pairs of addends that will give a sum of 8. (Use whole numbers.)

   (i) ___1___ and ___7___ (ii) ________ and ________

   (iii) ________ and ________ (iv) ________ and ________

4. In the number sentence 8 + 1 = 9, 8 and 1 are called the __________ and 9 is called the __________

Multiplication is the operation that assigns to two numbers a third number called the PRODUCT. The two numbers that are multiplied are called FACTORS. The operation of multiplication is indicated by placing a times sign (x) between the pair of numbers to be multiplied.

$$\underbrace{5 \times 7}_{\text{FACTORS}} = \underbrace{35}_{\text{PRODUCT}}$$

Fill in the blanks below.

1. In the number sentence $3 \times 6 = 18$, the factors are ________ and ________, and the product is ________. The symbol ________ is used to indicate that the operation is multiplication.

2. What is the product of 7 and 8? __________

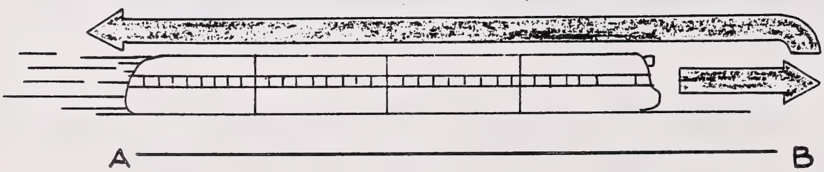
3. Give four different pairs of factors that will give a product of 64. (Use whole numbers.)

   (i) 1 and 64 (ii) ________ and ________

   (iii) ________ and ________ (iv) ________ and ________

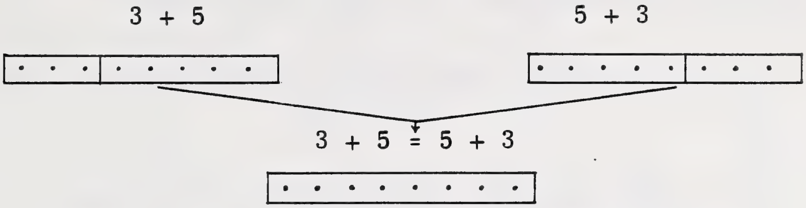
4. In the number sentence $25 \times 4 = 100$, 25 and 4 are called the __________ and 100 is called the __________.

B. Commutative Properties of Addition and Multiplication is Set W



Commutativity in mathematics can be compared with the behavior of a commuter train travelling between station A and station B. The train covers the same distance when it departs from station A and arrives at station B as when it departs from station B and arrives at station A. In mathematics, if we add or multiply any two whole numbers, the order in which we add or multiply the numbers does not affect the result.

For example, if we find the sum 3 + 5 and then find the sum 5 + 3, we will obtain the whole number 8 in each case.



The property which tells us that the order in which we add two whole numbers does not affect the sum is called the COMMUTATIVE PROPERTY OF ADDITION.

COMMUTATIVE PROPERTY OF ADDITION IN SET W
For any two whole numbers a and b,
$a + b = b + a$

The commutative property of addition can be used to check your answer to an addition problem. If you add the numbers in the reverse order, you should get the same result.

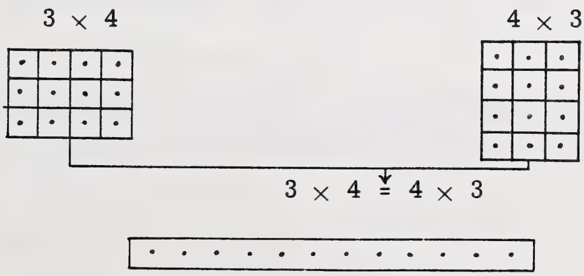
EXAMPLE: Add 8762 and 3568 in two different ways and check to see if you get the same result in both cases.

Solution

$$\begin{array}{r} 8762 \\ +3568 \\ \hline \end{array} \qquad \longleftarrow \boxed{\text{COMPLETE}} \longrightarrow \qquad \left.\begin{array}{r} 3568 \\ +8762 \\ \hline \end{array}\right\}$$

Did you get the same sum in both cases? ______

Similarly, the order in which we multiply two whole numbers does not affect the product. For example, if we find the product 3×4 and then find the product 4×3, we will obtain the whole number 12 in each case.



The property which tells us that the order in which we multiply two whole numbers does not affect the product is called the COMMUTATIVE PROPERTY OF MULTIPLICATION.

COMMUTATIVE PROPERTY OF MULTIPLICATION IN SET W
For any two whole numbers a and b,
$ab = ba$

(NOTE: ab means a×b)

The commutative property of multiplication can be used to check your answer to a multiplication problem. If you multiply the two numbers in the reverse order you should get same result.

EXAMPLE: Multiply 420 and 305 in two different ways and check to see if you get the same result in both cases.

Solution

$$\begin{array}{r} 420 \\ \times 305 \\ \hline \end{array} \qquad \longleftarrow \boxed{\text{COMPLETE}} \longrightarrow \qquad \begin{array}{r} 305 \\ \times 420 \\ \hline \end{array}$$

Did you get the same product in both cases?

____

Self-correcting Exercise #1

Answers to this exercise may be found on page 44 of this lesson.

1. State whether the commutative property of addition or the commutative property of multiplication justifies each of the following true statements.

(a) $12 + 13 = 13 + 12$ commutative property of addition.

(b) $7 \times 3 = 3 \times 7$ __________

(c) $7(6 + 9) = 7(9 + 6)$ __________

(d) $7(6 + 9) = (6 + 9)\,7$ __________

(e) $(8 \times 2) + 3 = 3 + (8 \times 2)$ __________

(f) $(8 \times 2) + 3 = (2 \times 8) + 3$ __________

(g) $ab + c = ba + c$ __________

(h) $ab + c = c + ab$ __________

C. Associative Properties of Addition and Multiplication in Set W.

"Which two of us should stick together first?" "It doesn't matter; we'll all be together by the end. Associate with anyone you want to."

The above diagram is a good reminder of the associative properties of addition and multiplication. In mathematics, if we add or multiply any three whole numbers, the manner in which we group the numbers does not affect the result.

Since addition and multiplication are binary operations on numbers, they involve only <u>two</u> numbers at a time. Therefore, we must define what we mean by the sum or product of <u>three</u> whole numbers.

For example, how would we compute the sum 5 + 7 + 3? Note that this problem can be approached in two ways:

<u>Method I</u>

Find the sum of the first two numbers and add this result to the third number.

$$
\begin{aligned}
&(5 + 7) + 3 \\
= \; &12 + 3 \\
= \; &15
\end{aligned}
$$

<u>Method II</u>

Find the sum of the last two numbers and add this result to the first number.

$$
\begin{aligned}
&5 + (7 + 3) \\
= \; &5 + 10 \\
= \; &15
\end{aligned}
$$

Note that the whole number 15 is obtained in both cases.

The property which tells us that in <u>adding</u> three whole numbers, the manner in which we <u>group</u> the numbers does not affect the sum is called the ASSOCIATIVE PROPERTY OF ADDITION.

ASSOCIATIVE PROPERTY OF ADDITION IN SET W

For any three whole numbers a, b, and c,

$$a + (b + c) = (a + b) + c$$

Note: According to the associative property, the three numbers are still written in the order a, b, c; only the grouping is changed. If the order were changed too, the commutative property would also apply.

Show that the following statements are true.

1. $(9 + 4) + 3 = 9 + (4 + 3)$

13+3	9+7
16	16

2. $(13 + 5) + 8 = 13 + (5 + 8)$

3. $7 + (7 + 4) = (7 + 7) + 4$

4. $16 + (4 + 10) = (16 + 4) + 10$

The associative property of addition can be used to simplify additions. Some addition questions are simplified considerably if we group addends to get multiples of 10.

EXAMPLE: $63 + 22 + 78 = \blacksquare$

Solution If we group 22 and 78, we will get a sum of 100, which is an easy number to work with.

$63 + 22 + 78$

$= 63 + (22 + 78)$ ← USE THE ASSOCIATIVE PROPERTY TO GROUP THESE TWO ADDENDS.

$= 63 + 100$ ← 100 IS A MULTIPLE OF 10.

$= 163$

Find the sum in each of the questions below. To permit easy addition, select groupings that will give multiples of 10.

1. $36 + 14 + 48$

   $= (36 + 14) + 48$

   $= \_\_\_\_\_ + \_\_\_\_\_$

   $= \_\_\_\_\_$

2. $25 + 28 + 32$

   $=$

3. $17 + 152 + 48$

   $=$

4. $149 + 11 + 38$

   $=$

Similarly, the manner in which we group three whole numbers in multiplication does not affect the product. For example, we could compute the product $2 \times 3 \times 4$ in two ways.

<u>Method I</u>	<u>Method II</u>
Find the product of the first two numbers and multiply this result by the third number.	Find the product of the last two numbers and multiply this result by the first number.
$(2 \times 3) \times 4$	$2 \times (3 \times 4)$
$= 6 \times 4$	$= 2 \times 12$
$= 24$	$= 24$

Note that the whole number 24 is obtained in both cases.

The property which tells us that in <u>multiplying</u> three whole numbers, the manner in which we <u>group</u> the numbers does not affect the product is called the ASSOCIATIVE PROPERTY OF MULTIPLICATION.

ASSOCIATIVE PROPERTY OF MULTIPLICATION IN SET W

For any three whole numbers a, b, and c,

$$a \times (b \times c) = (a \times b) \times c$$

(Again note that the numbers a, b, c are written in the same order; only the grouping changes.)

Show that the following statements are true.

1. $(13 \times 2) \times 7 = 13 \times (2 \times 7)$

   26×7 | 13×14

   182 | 182

2. $(6 \times 8) \times 5 = 6 \times (8 \times 5)$

3. $8 \times (8 \times 4) = (8 \times 8) \times 4$

4. $9 \times (7 \times 11) = (9 \times 7) \times 11$

The associative property of multiplication can be used to simplify multiplications. Some multiplication questions are simplified considerably if we group factors to obtain multiples of 100.

EXAMPLE: $16 \times 40 \times 5 = \square$

Solution If we multiply 40 and 5, we will get a product of 200 which is an easy number to work with.

$$16 \times 40 \times 5$$
$$= 16 \times (40 \times 5)$$
$$= 16 \times 200 \longleftarrow \text{MULTIPLE OF 100}$$
$$= 3200$$

Find the product in each question below. Select a grouping that simplifies your computation.

1. $13 \times 4 \times 250$
   $= 13 \times (4 \times 250)$
   $=$

2. $50 \times 2 \times 39$
   $=$

3. $25 \times 4 \times 15$
   $=$

4. $13 \times 8 \times 25$
   $=$

D. Comparison of Commutative and Associative Properties

1. The commutative properties allow us to change the order of writing adjacent numbers in a product or sum.
For example, the commutative property of addition allows us to make the following statements:

$5 + 7 = 7 + 5$ ← Order of writing 5 and 7 has been changed.
$3 + (2 + 6) = 3 + (6 + 2)$ ← Adjacent addends 2 and 6 have been interchanged.
$(m + n)s = (n + m)s$ ← Addends m and n have been switched.

The commutative property of multiplication justifies these statements:

$5 \times 7 = 7 \times 5$ ← Order of multiplying has been reversed.
$3 \times (2 \times 6) = 3 \times (6 \times 2)$ ← Adjacent factors 2 and 6 have been reversed.
$(m + n)s = s(m + n)$ ← $(m + n)$ and s are multiplied in a different order.

2. The associative properties allow us to change the manner of grouping numbers in a product or sum. For example, according to the associative property of addition:

$6 + (7 + 8) = (6 + 7) + 8$
$a + [(b + c) + d] = a + [b + (c + d)]$
← Numbers remain in the same position but are grouped differently.

The associative property of multiplication allows us to write:

$(6 \times 7) \times 8 = 6 \times (7 \times 8)$
$(3m)n = 3(mn)$
← Numbers remain in same position but are grouped differently.

3. The commutative and associative properties combined allow us to add or multiply a series of numbers in any manner we wish. For example, the product $4 \times 9 \times 25$ can most easily be determined if the factors 4 and 25 are multiplied together first. The commutative and associative properties combined allow us to rearrange and group the factors in this manner.

i.e.
$$\begin{aligned} 4 \times 9 \times 25 &= 4 \times 25 \times 9 && \text{(Commutative Property)} \\ &= (4 \times 25) \times 9 && \text{(Associative Property)} \\ &= 100 \times 9 \\ &= 900 \end{aligned}$$

NOTE: If you are trying to decide whether the commutative or associative property justifies a particular statement, check the terms or factors involved to see whether or not they appear in a different order. If the order has been changed, the commutative property applies. If the terms or factors appear in the same order and there has merely been a change in grouping, the associative property applies.

Find the sum or product in each question below. Use the commutative and associative properties to rearrange the terms or factors and select the best grouping for easy computation.

1. $113 + 25 + 7$

$= (\underline{113} + \underline{7}) + \underline{25}$

$= \underline{\quad} + \underline{\quad}$

$= \underline{\qquad}$

2. $20 \times 23 \times 5$

$=$

3. $36 + 48 + 12 + 64$

$= (\underline{\quad} + \underline{\quad}) + (\underline{\quad} + \underline{\quad})$

$= \underline{\quad} + \underline{\quad}$

$= \underline{\qquad}$

4. $15 \times 7 \times 2$

$=$

Self-correcting Exercise #2

Answers may be found on page 44 of this lesson.

1. Each of the following statements is true because of the commutative or associative properties of addition and multiplication. Name the property which justifies each statement. (Notes on page 9 should help you with this exercise.)

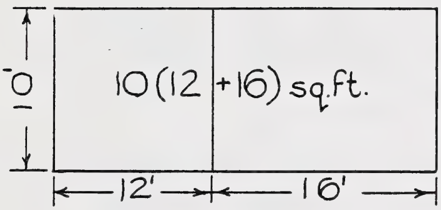
	Property
(a) $2a + 3b = 3b + 2a$	commutative property of addition
(b) $2(3 \times x) = (2 \times 3) \times x$	
(c) $(x + y)z = z(x + y)$	
(d) $m + 3n = 3n + m$	
(e) $r(ts) = r(st)$	
(f) $r(ts) = (rt)s$	
(g) $(m + n)(x + y) = (x + y)(m + n)$	
(h) $y(m + n) = y(n + m)$	
(i) $tr + ms = rt + sm$	
(j) $5 + (5 + 7) = (5 + 5) + 7$	
(k) $3(m - n) + 7 = 7 + 3(m - n)$	

E. The Distributive Property of Multiplication Over Addition in Set W

In everyday language, the word "distribute" means "to divide among several or many" or "to spread out so as to cover something." In mathematics the distributive property tells us how we can distribute a multiplier over a series of addends.

EXAMPLE: Suppose that a man is painting the walls of a corridor which is 10 feet high. He paints 12 feet of the corridor wall during the first hour and 16 feet of the wall during the next hour. What is the total area that he has painted?

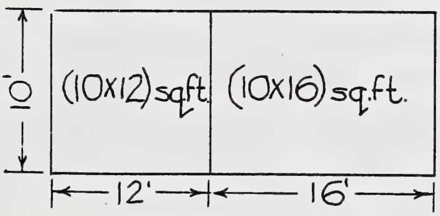
Solution



Method I

One way of finding the total area painted ignores the fact that some painting was done during one hour and some during the next. The total area is that of a rectangle (12 + 16) feet long and 10 feet high.

$$\begin{aligned} A &= 10(12 + 16) \\ &= 10(28) \\ &= \underline{\underline{280 \text{ sq. ft.}}} \end{aligned}$$



Method II

The total area painted could also be found by calculating the area painted each hour and finding the sum of these two areas. The total area is that of two rectangles $10' \times 12'$ and $10' \times 16'$.

$$\begin{aligned} A &= (10 \times 12) + (10 \times 16) \\ &= 120 + 160 \\ &= \underline{\underline{280 \text{ sq. ft.}}} \end{aligned}$$

Since we have arrived at the same result by both methods, we can conclude that:

$$10(12 + 16) = (10 \times 12) + (10 \times 16)$$

This means that the product of the factor 10 and the factor (12 + 16) can be determined by distributing the factor 10 over the two addends 12 and 16.

The property which tells us that a multiplier can be distributed over two addends is called the DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION.

DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION IN SET W

For any three whole numbers a, b, and c,

$$a(b + c) = (a \times b) + (a \times c)$$

The above property can be written in an alternative form. Since multiplication is commutative:

$a(b + c)$ can be written as $(b + c)a$.
$a \times b$ can be written as $b \times a$.
$a \times c$ can be written as $c \times a$.

Thus, the distributive property can take the following form:

For any three whole numbers a, b, and c,
$$(b + c)a = (b \times a) + (c \times a)$$

The distributive property can be extended to cover more than two addends. In general form, the distributive property could be written as follows:

For any whole numbers a, b, c, d, ...,
$$a(b + c + d + \ldots) = (a \times b) + (a \times c) + (a \times d) + \ldots$$

Like the commutative and associative properties, the distributive property can be used in simplifying computations. For example, if we wish to compute $(5 \times 97) + (5 \times 3)$, the straightforward way is to compute 5×97 and 5×3 and then add the results. An easier way is to observe that the expression $(5 \times 97) + (5 \times 3)$ is in the form $(a \times b) + (a \times c)$. Using the distributive property in reverse, we obtain:

$$\begin{aligned}(5 \times 97) + (5 \times 3) &= 5(97 + 3)\\ &= 5 \times 100\\ &= 500\end{aligned}$$

5 can be used as a common multiplier.

Note that both addends contain the factor 5.

Do the following computations by using the distributive property in reverse.

1. $(18 \times 3) + (2 \times 3) = (18 + 2)3 = (20)3 =$ _____

2. $(2 \times 98) + (2 \times 2) = 2(98 +$ ___ $) = 2($___$) =$ _____

3. $(6 \times 189) + (6 \times 11) = 6($___ $+$ ___$) = 6($___$) =$ _____

4. $(8 \times 96) + (8 \times 4) =$ ____________________

5. $(31 \times 4) + (9 \times 4) =$ ____________________

We use the distributive property in arithmetic whenever we multiply factors that have more than one digit.

EXAMPLE: 4 × 153 = ▨

Solution Break 153 into units, tens, and hundreds, and then apply the distributive property.

4 × 153 = 4(100 + 50 + 3)
= (4 × 100) + (4 × 50) + (4 × 3)
= _____ + _____ + _____
= _____

Multiply the following factors horizontally. First break up the 2 or 3 digit factors into units, tens, etc; and then apply the distributive property.

1. 7 × 19 = 7(10 + 9) = (7 × __) + (7 × __) = ___ + ___ = ___

2. 8 × 35 = 8(__ + _) = (8 × __) + (_ × __) = ___ + ___ = ___

3. 3 × 431 = 3(400 + 30 + 1) = (3 × ___) + (3 × ___) + (3 × ___)
= ___ + ___ + ___
= ___

The distributive property can also be used in simplifying algebraic expressions.

EXAMPLE: 4x + 3x + 9x = ▨

Solution We cannot really add letter expressions; we can only add numbers. If we apply the distributive property in reverse, we can simplify the given expression as follows:

4x + 3x + 9x
= (4 + 3 + 9)x ← distributive property (in reverse)
= 16x ← addition fact

Simplify each of the following expressions by using the distributive property in reverse.

1. 3a + 8a = (__ + __)a = ___a

2. 6x + 4x = (__ + __)x = ___x

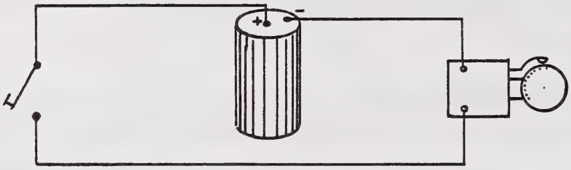
3. 17c + c = (__ + 1) c = ___c

4. a + 6a = ____________________

5. 7ab + 15ab = (__ + __)ab = ___ab

6. 19x + 15x + 3x = ____________________

F. Closure Properties of Addition and Multiplication in Set W



Closure in mathematics can be compared with the behavior of an electric circuit. When we close the key of an electric circuit, the circuit is closed and electrons will flow only within the system. In mathematics, if we add or multiply any two whole numbers, the result will always be another whole number.

For example, if we add the two whole numbers 101 and 32, we obtain the number 133. Note that 133 is also a whole number.

Add 10 and 55. _____ Is the sum a whole number? ________

Add 0 and 50. _____ Is the sum a whole number? ________

Add 615 and 1213. _____ Is the sum a whole number? ________

Can you choose a pair of whole numbers whose sum is <u>not</u> a whole number? __________

Since the sum of any two whole numbers is always a whole number, it is true that the set of whole numbers is <u>closed</u> under addition.

CLOSURE PROPERTY OF ADDITION IN SET W
For any whole numbers a and b, the sum of these numbers, $a + b$, is a whole number.

Similarly, when we <u>multiply</u> any two whole numbers, the result is a whole number. For example, if we multiply the two whole numbers 15 and 5, we obtain the number 75. Note that 75 is also a whole number.

Multiply 6 and 3. _____ Is the product a whole number? ________

Multiply 15 and 10. _____ Is the product a whole number? ________

Multiply 12 and 12. _____ Is the product a whole number? ________

Can you choose a pair of whole numbers whose product is <u>not</u> a whole number? __________ .

Since the product of any two whole numbers is always a whole number, it is true that the set of whole numbers is <u>closed</u> under multiplication.

CLOSURE PROPERTY OF MULTIPLICATION IN SET W
For any two whole numbers a and b, the product of these numbers, ab, is a whole number.

<u>EXERCISE - Properties of W Under Addition and Multiplication</u>

1. Fill in the blanks.

(a) For any whole numbers m and n, mn = nm because multiplication is ______________________.

(b) Operations which are performed on two numbers at a time are called ________________ operations.

(c) When we say that $3x + 7x = (3 + 7)x$, we are using the ______________________ property.

(d) According to the associative property of addition, $(3 + 5) + 7 =$ __________________.

(e) The sum of two whole numbers is always a __________ number.

(f) We know that $9 + 11 = 11 + 9$ because of the ________________ property of ______________________.

(g) According to the distributive property,

$$(a + b)c = (a \times c) + (\_\_ \times \_\_)$$

(h) Set W is closed under ________________ and ________________.

(i) $(35 \times 5) \times 7 = 35 \times (5 \times 7)$ because multiplication is

______________________.

(j) The ______________ and ______________ properties combined allow us to add or multiply a series of numbers in any manner we wish.

(k) $5(x + y) = 5x + 5y$ because ________________ distributes over
(multiplication or addition)
________________.
(multiplication or addition)

(l) When we multiply two numbers that have more than one digit, we use the ____________________ property.

2. Multiply. Then use the commutative property of multiplication to check your work in the space provided.

a. $\begin{array}{r} 135 \\ \underline{\times 27} \end{array}$ **Check** $\begin{array}{r} 27 \\ \underline{\times 135} \end{array}$

b. $\begin{array}{r} 506 \\ \underline{\times 702} \end{array}$ **Check**

3. Use the numbers 3, 7, and 11 to give an example of the distributive property.

______________ = ______________

4. Use the whole numbers 4 and 8 to illustrate the commutative properties of:

(a) addition

__________ = __________

(b) multiplication

__________ = __________

5. Simplify each of the following expressions by using the distributive property in reverse. (Express answers as single terms.)

(a) $4x + 13x =$ $(4+13)x = 17x$

(b) $3xy + 9xy + 17xy =$

(c) $3a + 5a + a =$

(d) $4xy + 4xy =$

(e) $2b + 3b + 5b + 9b =$

(f) $y + 5y + 2y =$

6. Name the property that justifies each statement. Choose from the following list.

LIST	
associative property of addition	distributive property
associative property of mult.	closure property of addition
commutative property of addition	closure property of mult.
commutative property of mult.	

(a) $(5 + 7)$ is a whole no. closure property of addition

(b) $(3a)b = 3(ab)$ ______

(c) $5a + 5b = 5(a + b)$ distributive property

(d) $(x + y) + z = x + (y + z)$ ______

(e) $6 \times 7 = 7 \times 6$ ______

(f) (3×5) is a whole no. ______

(g) $6 + 7y = 7y + 6$ ______

(h) $(4 \times 5) + 3 = 3 + (4 \times 5)$ ______

(i) $5 + (6 \times 7) = 5 + (7 \times 6)$ ______

(j) $(6 \times 7) \times 30 = 6 \times (7 \times 30)$ ______

(k) $(6 \times 7) \times 30 = (7 \times 6) \times 30$ ______

(l) $(p + q)s = ps + qs$ ______

(m) $(a + b)c = c(a + b)$ ______

(n) $3x + 2x = (3 + 2)x$ ______

(o) $(3 \times 5) + (7 \times 2) = (7 \times 2) + (3 \times 5)$ ______

(p) $(6 \times 7) \in W$ ______

(q) $3(ab) = 3(ba)$ ______

(r) $6(xy) = (6x)y$ ______

Topic Two: Checking Properties for Subtraction in Set W

A. The Operation of Subtraction

Subtraction is another fundamental operation of arithmetic. Like addition and multiplication, it is a binary operation because it is performed on two numbers at a time.

Subtraction is the operation that assigns to two numbers a third number called the DIFFERENCE. The number that you subtract from is called the MINUEND and the number you subtract is called the SUBTRAHEND. The operation of subtraction is indicated by placing a minus sign (-) between the pair of number to be subtracted.

EXAMPLE: 7 - 5 = 2 ← DIFFERENCE
MINUEND ↗ ← SUBTRAHEND

In the set of whole numbers, the operation of subtraction is only defined if the minuend is greater than the subtrahend. For example, there is no whole number n that satisfies the condition 5 - 7 = n.

Fill in the blanks below.

1. In the number sentence 9 - 3 = 6, the minuend is _____, the subtrahend is _____, and the difference is _____. The symbol __________ is used to indicate that the operation is subtraction.

2. What is the difference of 10 and 7? __________

3. Fill in the blanks with pairs of whole numbers that will give a difference of 2.

   (i) 15 - 13 = 2 (ii) _____ - _____ = 2

   (iii) _____ - _____ = 2 (iv) _____ - _____ = 2

4. In the number sentence 12 - 5 = 7, 12 is called the _____________, 5 is called the _____________, and 7 is called the _____________.

5. A subtraction is possible in set W only if the ________________ is larger than the ________________.

An INVERSE OPERATION is any operation which undoes whatever has been done. For example, if a door is opened and then closed we will both start and finish with the door being closed. We can say that "closing" is the inverse operation of "opening".

Suppose that we are given a certain number and first perform the operation of adding 5 to it and then perform the operation of subtracting 5. What number do we end up with?

Given Number	Add 5	Subtract 5	Resulting Number
2	$2 + 5 = 7$	$7 - 5 = 2$	2
5	$5 + 5 = 10$	$10 - 5 = 5$	5
11	$11 + 5 = 16$	$16 - 5 = 11$	11
39	$39 + 5 = 44$	$44 - 5 = 39$	39

(Note that in each case you start and finish with the same number.)

We can conclude that "adding 5" and "subtracting 5" are inverse operations. In general, we can say that:

SUBTRACTION IS THE INVERSE OPERATION OF ADDITION.

Fill in the blanks with the missing numbers.

(i) $(8 + 7) - 7 =$ ____

(ii) $(15 - 9) + 9 =$ ____

(iii) $(17 - 5) +$ ____ $= 17$

(iv) $(14 + 3) -$ ____ $= 14$

(v) $(16 -$ ____ $) + 8 = 16$

(vi) $($____ $- 7) + 7 = 21$

(vii) $($____ $+ 6) - 6 = 35$

(viii) $(15 + 12) -$ ____ $= 15$

Since subtraction is the inverse operation of addition, we can define subtraction in terms of addition. Every subtraction question can be rewritten as an addition question. For example, in order to answer the question:

$$8 - 5 = \boxed{},$$

we may ask ourselves, "What number must be added to 5 to give a sum of 8?" Thus we could change this subtraction question to the addition question:

$$5 + \boxed{} = 8.$$

From our addition tables, we know that the missing addend must be 3.

Completing sentences such as 5 + ▨ = 8 may be called "finding the missing addend." Thus, subtraction may be referred to as the operation of finding the missing addend. In the case of addition, we seek a sum of two given addends while in the case of subtraction, we seek one of the addends of a given sum.

DEFINITION OF SUBTRACTION IN SET W
For any three whole numbers a, b, and c;
$a - b = c$ if and only if $a = b + c$
"Subtract b from a" means "find the number c which must be added to b to make a".

Complete the following statements.

1. 7 - 2 means "find the number which must be added to ______ to make 7."

2. 125 - 79 means "find the number which must be added to ______ to make ______."

3. 23 - 6 means "find ____ ____"

4. 18 - 6 = ______ because 6 + ___ = 18.

5. 26 - 9 = ______ because ___ + ___ = 26.

Write an addition sentence for each of the following subtraction sentences.

1. 12 - 9 = 3   9 + 3 = 12

2. 7 - 6 = 1   ______________

3. 36 - 17 = 19   ______________

4. 37 - 23 = x   23 + x = 37

5. 42 - 35 = y   ______________

6. 16 - 10 = n   ______________

B. <u>Does the Commutative Property Hold Under Subtraction in Set W?</u>

If there were a commutative property of subtraction, this would mean that $a - b = b - a$ for any whole numbers a and b. Let us substitute some values for a and b and see if this property holds.

EXAMPLE: Is it true that $5 - 3 = 3 - 5$?
<u>Solution</u>

We know that $5 - 3 = 2$, since 2 is the number which must be added to 3 to make 5. But, there is no <u>whole</u> number which satisfies the condition $3 - 5 = \square$, since there is no whole number that we can add to 5 to make 3. Thus, $5 - 3 = 3 - 5$ is a <u>false</u> statement.

SYMBOL FOR "DOES NOT EQUAL"

From the example above, we know that $a - b \neq b - a$ for all whole numbers a and b. Thus, the order in which we subtract two whole numbers <u>does</u> affect the result.

SUBTRACTION IS <u>NOT</u> COMMUTATIVE IN THE SET OF WHOLE NUMBERS

Is it true that:

(i) $8 - 9 = 9 - 8$? _____ (ii) $20 - 8 = 8 - 20$? _____

(iii) $7 - 2 = 2 - 7$? _____ (iv) $100 - 3 = 3 - 100$? _____

C. <u>Does the Associative Property Hold Under Subtraction in Set W?</u>

If there were an associative property of subtraction, it would be true that $a - (b - c) = (a - b) - c$ for any whole numbers a, b, and c. Let us substitute some values for a, b, and c, and see if this property holds.

EXAMPLE: Is it true that $9 - (7 - 1) = (9 - 7) - 1$?
<u>Solution</u> Fill in the blanks below.

$9 - (7 - 1)$	$(9 - 7) - 1$
$= 9 -$ _____	$=$ _____ $- 1$
$=$ _____	$=$ _____

Are the two sides equal? _____

From the example at the bottom of page 21, we know that $a - (b - c) \neq (a - b) - c$ for all whole numbers a, b, and c. Thus, the manner of grouping three numbers in subtraction <u>does</u> affect the difference.

SUBTRACTION IS <u>NOT</u> ASSOCIATIVE IN THE SET OF WHOLE NUMBERS.

D. Does Multiplication Distribute Over Subtraction in Set W?

If there were a distributive property of multiplication over subtraction in set W, this would mean that $a(b - c) = ab - ac$ for any whole numbers a and b.

EXAMPLE 1: Is it true that $5(6 - 3) = (5 \times 6) - (5 \times 3)$?
<u>Solution</u> Fill in the blanks below.

$5(6 - 3)$	$(5 \times 6) - (5 \times 3)$
$= 5 \times$ ____	$=$ ____ $-$ ____
$=$ ____	$=$ ____

Are the two sides equal? ________

EXAMPLE 2: Is it true that $2(3 - 4) = (2 \times 3) - (2 \times 4)$?
<u>Solution</u>

$2(3 - 4)$	$(2 \times 3) - (2 \times 4)$
	$= 6 - 8$
We cannot work out $(3 - 4)$ in Set W since the answer is not a whole number.	We cannot work out $(6 - 8)$ in set W since the answer is not a whole number.

In general, multiplication distributes over subtraction in set W as long as the subtraction yields a whole number.

MULTIPLICATION DISTRIBUTES OVER SUBTRACTION
For any whole numbers a, b, c, (where $b > c$)
$a(b-c) = ab - ac$

Multiply the following expressions by using the distributive property.

1. $7(a - b + c) =$ $7a - 7b + 7c$
2. $2(x - y) =$ ______
3. $3(a + b + c) =$ ______
4. $6(a + b - c) =$ ______
5. $5(x - y - z) =$ ______

Do the following computations by using the distributive property in reverse.

1. $(6 \times 17) - (6 \times 12) =$ $6(17-12) = 6(5) = 30$
2. $(8 \times 7) - (3 \times 7) =$ ______
3. $(4 \times 16) - (2 \times 16) =$ (___ - ___) 16 = ______
4. $(40 \times 7) - (40 \times 5) =$ ______
5. $(8 \times 23) - (7 \times 23) =$ ______

Simplify each of the following expressions by using the distributive property in reverse.

1. $6a + 2a - a =$ $(6+2-1)a = 7a$
2. $14x - 7x =$ ______
3. $35ab - 25ab =$ ______
4. $7x^2 + 3x^2 - 2x^2 =$ (___ + ___ - ___) x^2 = ______
5. $13y - 7y - 5y =$ (___ - ___ - ___) y = ______

E. Is Set W Closed Under Subtraction?

Is the difference of two whole numbers always a whole number? Write W after each of the indicated subtractions where the result is a whole number.

1. 16 - 9 W	4. 62 - 70 _____
2. 27 - 35 _____	5. 33 - 29 _____
3. 17 - 17 _____	6. 6 - 9 _____

The result is not a whole number in the above questions numbered _____, _____, and _____. Thus, the result obtained when two whole numbers are subtracted is not always a whole number.

SET W IS NOT
CLOSED UNDER SUBTRACTION

Self-correcting Exercise #3

Answers may be found on page 44 of this lesson.

1. Why is each of the following statements false?

(a) $8 - (6 - 2) = (8 - 6) - 2$ Subtraction is not associative.

(b) $6 - 3 = 3 - 6$ _____

(c) $(3 - 6)$ is a whole number. _____

(d) $5(x - y) = 5(y - x)$ _____

(e) $x - (y - z) = (x - y) - z$ _____

(f) $(a - b) - c = c - (a - b)$ _____

2. Why is each of the following statements true?

(a) $x + 3 = 3 + x$ Addition is commutative.

(b) $5 + (7 + 9) = (5 + 7) + 9$ _____

(c) $(10 + 11)$ is a whole number. _____

(d) $a(b - c) = (b - c)\,a$ ______________________

(e) $x + (y - z) = (y - z) + x$ ______________________

(f) $(6 \times 3) \in W$ ______________________

(g) $[(a + b)(c + d)](e + f) = (a + b)[(c + d)(e + f)]$

____

Topic Three: Checking Properties for Division in Set W

A. The Operation of Division

Division is a binary operation that assigns to two numbers a third number called the QUOTIENT. The number you divide by is called the DIVISOR and the number you divide into is called the DIVIDEND. The operation of division is indicated by placing a division symbol (÷) between the pair of numbers to be divided. (Remember that the divisor is always written after the division symbol.)

QUOTIENT

EXAMPLE: $8 \div 2 = 4$

DIVIDEND DIVISOR

In the set of whole numbers, the operation of division is only defined if the divisor is smaller than the dividend and goes evenly into it. For example, there is no whole number n that satisfies the condition $7 \div 4 = n$.

Fill in the blanks below.

1. In the number sentence $15 \div 3 = 5$, the divisor is ________, the dividend is ________, and the quotient is ________. The symbol ________ is used to indicate that the operation is division.

2. What is the quotient of 36 and 9? _____

3. Fill in the blanks with pairs of numbers that will give a quotient of 3.

   (i) 27 ÷ 9 = 3 (ii) _____ ÷ _____ = 3

   (iii) _____ ÷ _____ = 3 (iv) _____ ÷ _____ = 3

4. In the number sentence $24 \div 6 = 4$, 24 is called the ______________, 6 is called the ______________, and 4 is called the ______________.

Suppose that we are given a certain number and first perform the operation of multiplying it by 5 and then perform the operation of dividing by 5. What number do we end up with?

Given Number	Multiply by 5	Divide by 5	Resulting Number
5	$5 \times 5 = 25$	$25 \div 5 = 5$	5
7	$7 \times 5 = 35$	$35 \div 5 = 7$	7
42	$42 \times 5 = 210$	$210 \div 5 = 42$	42
65	$65 \times 5 = 325$	$325 \div 5 = 65$	65

(Note that the given number and resulting number are the same.)

We can conclude that "multiplying by 5" and "dividing by 5" are inverse operations. In general, we can say that:

DIVISION IS THE INVERSE OPERATION OF MULTIPLICATION.

Fill in the blanks with the missing numbers.

(i) $(6 \div 2) \times 2 = \underline{\qquad}$

(ii) $(6 \times 2) \div 2 = \underline{\qquad}$

(iii) $(8 \div 4) \times \underline{\qquad} = 8$

(iv) $(32 \times 4) \div \underline{\qquad} = 32$

(v) $(12 \div \underline{\quad}) \times 3 = 12$

(vi) $(\underline{\quad} \div 3) \times 3 = 24$

(vii) $(10 \times \underline{\quad}) \div 5 = 10$

(viii) $(2 \times 3) \div \underline{\qquad} = 2$

Since division is the inverse operation of multiplication, we can define division in terms of multiplication. Every division question can be rewritten as a multiplication question. For example, in order to answer the question

$$20 \div 5 = \boxed{},$$

we may ask ourselves, "What number must we multiply 5 by to get a product of 20?" Thus, we could change this division question to the multiplication question

$$5 \times \boxed{} = 20.$$

From our multiplication tables, we know that the missing factor must be 4.

Completing sentences such as $5 \times \square = 20$ may be called "finding the missing factor." Thus, division may be referred to as the operation of finding the missing factor. In multiplication, we seek a product of two given factors while in division, we seek one of the factors of a given product.

DEFINITION OF DIVISION IN SET W
For any three whole numbers a, b, and c;
$a \div b = c$ if and only if $a = b \times c$.
"Divide a by b" means "find the number which b must be multiplied by to make a."

Complete the following statements.

1. $12 \div 3$ means "find the number which must be multiplied by __________ to make 12."

2. $125 \div 25$ means "find the number which must be multiplied by __________ to make __________."

3. $b \div a$ means "find the ____."

4. $44 \div 11 =$ __________ because $11 \times$ ___ $= 44$.

5. $63 \div 7 =$ __________ because __ $\times$ ___ $= 63$.

Write a multiplication sentence for each of the following division sentences.

1. $42 \div 7 = 6$ $\quad$ $7 \times 6 = 42$
2. $90 \div 9 = 10$ $\quad$ ____________
3. $32 \div 8 = 4$ $\quad$ ____________
4. $12 \div 4 = x$ $\quad$ $4 \times x = 12$
5. $14 \div 7 = n$ $\quad$ ____________
6. $32 \div 4 = y$ $\quad$ ____________

B. Does the Commutative Property Hold Under Division in Set W?

If there were a commutative property of division, this would mean that $a \div b = b \div a$ for any whole numbers a and b. Let us substitute some values for a and b and see if this property holds.

EXAMPLE: Is it true that $6 \div 3 = 3 \div 6$?
Solution We know that $6 \div 3 = 2$ because 2 is the number which we can multiply by 3 to make 6. But, there is no whole number which satisfies the condition $3 \div 6 = \boxed{}$, since there is no whole number we can multiply by 6 to make 3.
Thus, $6 \div 3 = 3 \div 6$ is a false statement.

From the example, we know that $a \div b \neq b \div a$ for all whole numbers a and b. Thus the order in which we divide two whole numbers does affect the result.

DIVISION IS NOT COMMUTATIVE IN THE SET OF WHOLE NUMBERS.

Is it true that

(i) $18 \div 6 = 6 \div 18$? _____

(ii) $4 \div 2 = 2 \div 4$? _____

(iii) $35 \div 7 = 7 \div 35$? _____

C. Does the Associative Property Hold Under Division in Set W?

If there were an associative property of division, it would be true that $a \div (b \div c) = (a \div b) \div c$ for any whole numbers a, b, and c.

EXAMPLE: Is it true that $36 \div (6 \div 3) = (36 \div 6) \div 3$?
Solution

Fill in the blanks below.

$36 \div (6 \div 3)$	$(36 \div 6) \div 3$
$= 36 \div$ _____	$=$ _____ $\div 3$
$=$ _____	$=$ _____

Are the two sides equal? __________

From the example, we know that $a \div (b \div c) \neq (a \div b) \div c$ for all whole numbers a, b, and c. Thus the manner of grouping three numbers in division <u>does</u> affect the quotient.

DIVISION IS <u>NOT</u> ASSOCIATIVE IN THE SET OF WHOLE NUMBERS.

D. <u>Is Set W Closed Under Division?</u>

Is the quotient of two whole numbers always a whole number? Write W after each of the indicated divisions where the result <u>is</u> a whole number.

1. $16 \div 9$ _____
2. $45 \div 5$ W
3. $14 \div 4$ _____
4. $18 \div 9$ _____
5. $3 \div 7$ _____
6. $32 \div 8$ _____

The result is <u>not</u> a whole number in the above questions numbered _____, _____, and _____. Thus the result obtained when two whole numbers are divided is <u>not</u> always a whole number.

SET W IS <u>NOT</u> CLOSED UNDER DIVISION.

<u>Self-correcting Exercise #4</u>

Answers may be found on page 45 of this lesson.

1. Why is each of the following statements false?

(a) $(8 \div 4) \div 2 = 8 \div (4 \div 2)$ ____________________

(b) $(8 \div 6) \in W$ ____________________

(c) $(6 \times 2) \div 4 = 4 \div (6 \times 2)$ ____________________

(d) $12 \div 6 = 6 \div 12$ ____________________

(e) $a + (b \div c) = a + (c \div b)$ ____________________

(f) $x \div (y \div z) = (x \div y) \div z$ ____________________

EXERCISE - Properties of W Under the Four Fundamental Operations

1. Fill in the blanks.

(a) Set W is not closed under ________________ and ______________.

(b) The inverse operation of multiplication is ____________________.

(c) "Subtract 4 from 10" means "find the number which must be added to _____ to make _____."

(d) 15 - (5 - 2) $\neq$ (15 - 5) - 2 because subtraction is not ________________________________.

(e) The inverse operation of subtraction is ____________________.

(f) "Divide 6 by 3" means "find the number which must be multiplied by _____ to make _____."

(g) (6 - 3) does not equal (3 - 6) because ________________ is not commutative.

(h) 2 ÷ 8 is not a whole number because set W is not ______________________ under ______________________.

(i) 5(x - y) = 5x - 5y because ________________ distributes over
(multiplication or subtraction)
________________ whenever the subtraction yields a
(multiplication or subtraction)
whole number.

2. Decide whether each of the following statements is true or false. Give a reason for your answer. (Review self-correcting exercises on pages 24 and 29.)

(a) (2 - 4) is a whole number. false

Set W is not closed under subtraction.

(b) xy = yx ________

Multiplication is commutative.

(c) 3 + (m + n) = (3 + m) + n ________

____

(d) $8 - (3 - 2) = (8 - 3) - 2$ ________

____

(e) $(7 \times 5) - 1 = 1 - (7 \times 5)$ ________

____

(f) (8×3) is a whole number. ________

____

(g) $6 \div 2 = 2 \div 6$ ________

____

(h) $12 \div (6 \div 2) = (12 \div 6) \div 2$ *false*

Division is not associative.

(i) $3(2 - 5)$ is a whole number. ________

____

(j) $3x + 5yx + 2xz = (3 + 5y + 2z)x$ ________

____

(k) $(a + b)(c + d) = (c + d)(a + b)$ ________

____

(l) $(8 \div 2) + 5 = 5 + (8 \div 2)$ ________

____

(m) $5(7 - 3) = 5(3 - 7)$ ________

____

(n) $2x - (x - y) = (2x - x) - y$ ________

____

(o) $(5x)y = 5(xy)$ ________

____

(p) $3(a + b + c) = 3a + 3b + 3c$ ________

____

3. Decide whether or not the results of the following operations are whole numbers. Write "yes" in the blank if they are, and "no" if they aren't.

(a) $5 + 7$ yes	(e) 9×7 ____	(i) $4 - 8$ ____
(b) $5 - 7$ ____	(f) 7×9 ____	(j) $4 + 8$ ____
(c) $7 - 5$ ____	(g) $9 \div 3$ ____	(k) $6 - 6$ ____
(d) $3 \div 6$ ____	(h) $9 \div 4$ ____	(l) $4 \div 9$ ____

4. Look at the sets below and decide if they are closed under addition. (Add pairs of elements from the set. If the results always belong to the given set, the set is closed under addition.)

(a) $O = \{1, 3, 5, \ldots\}$

$1 + 3 = 4$
$1 + 7 = 8$
$3 + 9 = 12$

Are the results always members of O? no Why? The results are even numbers while set O contains only odd numbers.

Is set O closed under addition? no

(b) $E = \{2, 4, 6, \ldots\}$

$2 + 6 =$ ____
$14 + 6 =$ ____
$12 + 28 =$ ____

Are the results always members of E? ____ Why? ____________

Is set E closed under addition? ____

(c) $D = \{5, 6, 7, \ldots, 20\}$

$5 + 6 =$ ____
$7 + 16 =$ ____
$18 + 19 =$ ____

Are the results always members of D? ____ Why? ____________

Is set D closed under addition? ____

(d) $B = \{5, 10, 15, \ldots\}$

$5 + 10 =$ ____
$10 + 15 =$ ____
$20 + 25 =$ ____

Are the results always members of B? ____ Why? ____________

Is set B closed under addition? ____

5. What is the opposite of each of the following operations?

	Opposite
(a) Opening the door	__________
(b) Putting on your shoes	__________
(c) Subtracting 8	__________
(d) Turning on the tap	__________
(e) Multiplying by 7	__________
(f) Dividing by 4	__________

6. To check subtraction problems, we add. If we subtract 5 from 12 and get 7, we check by adding 7 and 5 to see if we get 12. Do the following subtractions and check by adding.

(a) $\begin{array}{r} 17 \\ -3 \\ \hline 14 \end{array}$ Check $\begin{array}{r} 3 \\ +14 \\ \hline 17 \end{array}$

(b) $\begin{array}{r} 356 \\ -127 \\ \hline \end{array}$ Check

(c) $\begin{array}{r} 60005 \\ -49898 \\ \hline \end{array}$ Check

(d) $\begin{array}{r} 40,306 \\ -39,428 \\ \hline \end{array}$ Check

7. To check division problems, we multiply. If we divide 15 by 3 and get 5, we check by multiplying 3 and 5 to see if we get 15.

(a) $12 \overline{)1008}$ = 84

$\begin{array}{r} 96 \\ \hline 48 \\ 48 \\ \hline 0 \end{array}$

Check $\begin{array}{r} 84 \\ \times 12 \\ \hline 168 \\ 84 \\ \hline 1008 \end{array}$

(b) $42 \overline{)16716}$ Check

(c) $412 \overline{)8652}$ Check

(d) $19 \overline{)76038}$ Check

8. Find the whole number n which makes each sentence true. If such a number does not exist, write "no whole number" in the blank.

(a) $12 - 8 = n$

$n =$ ________

(b) $8 - 15 = n$

$n =$ no whole number

(c) $4 \times n = 24$

$n =$ ________

(d) $n - 2 = 5$

$n =$ ________

(e) $23 - n = 23$

$n =$ ________

(f) $n \div 3 = 4$

$n =$ ________

(g) $21 - n = 23$

$n =$ ________

(h) $17 \div n = 6$

$n =$ ________

(i) $18 - n = 12$

$n =$ ________

(j) $216 \div n = 6$

$n =$ ________

Topic Four: Special Properties of Zero

The element zero which belongs to the set of whole numbers $W = \{0, 1, 2, 3, \ldots\}$ has some unique properties with respect to the operations of addition, subtraction, multiplication, and division.

A. Addition

When we add zero to any whole number, that number is left unchanged. Because zero possesses this special property, it is called the IDENTITY ELEMENT FOR ADDITION or the ADDITIVE IDENTITY.

> For any whole number a,
> $a + 0 = a$
> $0 + a = a$

Thus,

$0 + 1 = 1$

$12 + 0 = 12$

$100 + 0 = 100$

$0 + 7 = 7$

NUMBER IS UNCHANGED WHEN 0 IS ADDED.

Fill in the blanks below.

$18 + 0 =$ ___	$0 + 17 =$ ___
___ $+ 13 = 13$	$15 +$ ___ $= 15$
$43 + 0 =$ ___	___ $+ 0 = 44$
$0 +$ ___ $= 8$	$0 +$ ___ $= 200$

B. Subtraction

(i) Subtracting zero from any whole number leaves the number unchanged.

For any whole number a, $a - 0 = a$

Thus,

$0 - 0 = 0$

$1 - 0 = 1$

$13 - 0 = 13$

$100 - 0 = 100$

NUMBER IS UNCHANGED.

Fill in the blanks below.

$23 - 0 =$ ___ $14 -$ ___ $= 14$

$16 -$ ___ $= 16$ $32 - 0 =$ ___

$85 - 0 =$ ___ ___ $- 0 =$ ___

___ $- 0 = 4$ $2 -$ ___ $= 2$

(ii) The result of subtracting two identical whole numbers is zero.

For any whole number a, $a - a = 0$

Thus,

$1 - 1 = 0$
$5 - 5 = 0$
$23 - 23 = 0$
$9 - 9 = 0$

RESULT IS 0.

Fill in the blanks below.

$7 - 7 =$ ___ $12 -$ ___ $= 0$
$8 -$ ___ $= 0$ ___ $- 33 = 0$
___ $- 3 = 0$ $4 -$ ___ $= 0$
$16 - 16 =$ ___ $112 - 112 =$ ___

C. Multiplication

Multiplying any number by zero gives a product of zero. (In general, when zero appears as a factor, the product is zero.)

For any whole number a, $a \times 0 = 0$, $0 \times a = 0$

Thus,

$0 \times 0 = 0$

$0 \times 8 = 0$

$17 \times 0 = 0$

$7 \times 0 \times 6 \times 5 = 0$

RESULT IS 0.

Fill in the blanks below.

$0 \times 12 =$ ___ ___ $\times 35 = 0$

___ $\times 7 = 0$ $6 \times$ ___ $= 0$

$4 \times$ ___ $= 0$ $2 \times$ ___ $\times 3 = 0$

$3 \times 0 \times 8 =$ ___ $5 \times 8 \times$ ___ $= 0$

D. Division

Zero as the Dividend

What do you think the answer is to the question $0 \div 5 = \square$?

Remember that in answering questions of the type $24 \div 4 = \square$, we asked outselves, "What number must we multiply 4 by to give a product of 24?" Thus, we could change the division question $24 \div 4 = \square$ to the multiplication question $4 \times \square = 24$.

We should be able to use this same method of approach in finding an answer to the question $0 \div 5 = \square$. We can ask ourselves, "What number must we multiply 5 by to give a product of 0?" That is, we must solve the following multiplication question:

$$5 \times \square = 0.$$

We know from the multiplication properties of zero, that a product can be zero if one of the factors is zero. Therefore, the missing factor is zero.

i.e. $5 \times \boxed{0} = 0$

Changing back to a division question, we can say that

$$0 \div 5 = \boxed{0}$$

In general, whenever zero is the dividend (the number you are dividing into) and the divisor is a non-zero number, the quotient (or answer) is zero.

For any whole number a other than zero, $0 \div a = 0$

DIVIDEND DIVISOR QUOTIENT

Thus,

$0 \div 1 = 0$

$0 \div 7 = 0$

$0 \div 13 = 0$

DIVIDEND IS 0. QUOTIENT IS 0.

Fill in the blanks below.

$0 \div 9 =$ ___ ___ $\div 4 = 0$

$0 \div 2 =$ ___ $0 \div$ ___ $= 0$

___ $\div 3 = 0$ $0 \div$ ___ $= 0$

Zero as the Divisor

What do you think the answer is to the question $5 \div 0 = \square$?

In finding a solution to this question, we can ask ourselves, "What number must we multiply zero by to give a product of 5?" That is, we must solve the following multiplication question:

$$0 \times \square = 5$$

But if the product of two numbers is <u>not</u> zero (in this case it is 5), then neither of the numbers can be zero. We have run into a contradiction here since one of the numbers <u>is</u> zero. There is no whole number than can be used to fill <u>in</u> the blank in the above statement.

$$0 \times \begin{pmatrix}\text{no whole} \\ \text{number}\end{pmatrix} = 5.$$

Changing back to a division question, we can say that

$$5 \div 0 = (\text{no whole number}).$$

Therefore, the expression $5 \div 0$ is undefined.

In general, whenever zero appears as a divisor, we say that the quotient is <u>undefined</u>.

For any whole number a, $a \div 0$ is undefined

DIVIDEND DIVISOR

Thus,

$1 \div 0$ is undefined.

$0 \div 0$ is undefined.

$27 \div 0$ is undefined.

DIVISOR IS 0. QUOTIENT IS UNDEFINED.

Fill in the blanks below.

$15 \div$ _____ is undefined.

$4 \div$ _____ is undefined.

$13 \div 0$ is __________.

___ $\div 0$ is undefined.

Self-correcting Exercise #5

Answers may be found on page 45 of this lesson.

1. Decide if each of the following expressions "equals zero" or is "undefined".

(a) $3 \div 0$ undefined

(b) $0 \div 3$ ______

(c) 0×5 ______

(d) $6 \times 0 \times 3$ ______

(e) $(8 - 8) \div 2$ ______

(f) $5(9 - 9)$ ______

(g) $4 \div (3 \times 0)$ ______

(h) $(3 + 6) \div 0$ ______

(i) $0 \div (3 + 0)$ ______

(j) $4 \times 2 \times 0$ ______

2. Simplify each expression. (Some will be "undefined".)

(a) $0 \times 3 =$

(b) $5 + 0 =$

(c) $8 - 0 =$

(d) $12 \div 0 =$

(e) $0 \div 7 =$

(f) $12 \div (6 - 6) = 12 \div$ ___ $=$

(g) $(2 \times 0) + 0 =$ ___ $+ 0 =$

(h) $0 \times 9 \times 10 =$

(i) $(9 + 0)2 =$ ___ $\times 2 =$

(j) $(0 \div 4)2 =$ ___ $\times 2 =$

(k) $0(1 + 2 + 3) = 0 \times$ ___ $=$

(l) $(0 \times 5) \div 16 =$ ___ $\div 16 =$

(m) $0 \div (3 - 0) = 0 \div$ ___ $=$

(n) $15 \div (3 + 0) = 15 \div$ ___ $=$

Topic Five: Special Properties of One

The whole number "one" also has some special properties under the operations of multiplication and division.

A. Multiplication

When we multiply any whole number by one, that number is left unchanged. Because "one" possesses this special property, it is called the IDENTITY ELEMENT FOR MULTIPLICATION or the MULTIPLICATIVE IDENTITY.

$$\boxed{\begin{array}{c}\text{For any whole number } a,\\ a \times 1 = a\\ 1 \times a = a\end{array}}$$

Thus,

Thus	Fill in the blanks below.	
$1 \times 6 = 6$	$1 \times 12 = \_\_\_$	$5 \times \_\_ = 5$
$7 \times 1 = 7$	$4 \times 1 = \_\_\_$	$1 \times \_\_ = 14$
$1 \times 23 = 23$	$\_\_ \times 3 = 3$	$\_\_ \times 1 = 52$
$1 \times 2 = 2$	$\_\_ \times 1 = 100$	$1 \times \_\_ = 0$

B. Division

(i) When any whole number is divided by one, that number is left unchanged.

$$\boxed{\begin{array}{c}\text{For any whole number } a,\\ a \div 1 = a\end{array}}$$

Thus,

Thus	Fill in the blanks below.	
$6 \div 1 = 6$	$7 \div 1 = \_\_\_$	$\_\_ \div 1 = 0$
$4 \div 1 = 4$	$12 \div \_ = 12$	$5 \div \_\_ = 5$
$17 \div 1 = 17$	$\_\_ \div 1 = 3$	$2 \div 1 = \_\_\_$

(ii) The result of dividing two identical whole numbers is "one".

For any whole number a other than zero, $a \div a = 1$

Remember that $0 \div 0$ is undefined.

Thus,

$6 \div 6 = 1$

$5 \div 5 = 1$

$1 \div 1 = 1$

Fill in the blanks below.

$12 \div 12 =$ ___	___ $\div 3 = 1$
$2 \div 2 =$ ___	$100 \div$ __ $= 1$
$7 \div$ __ $= 1$	$9 \div 9 =$ ___

Self-correcting Exercise #6

Answers may be found on page 46 of this lesson.

1. Simplify the following expressions.

(a) $8 \div 8 =$

(b) $1 \times 64 =$

(c) $8 \div 1 =$

(d) $1 \times 0 =$

(e) $0 + (4 \div 1) = 0 +$ ___ $=$

(f) $1 \times (3 \div 3) = 1 \times$ ___ $=$

(g) $(5 \div 5) \div 0 =$ ___ $\div 0 =$

(h) $(6 \div 6) + 0 =$ ___ $+ 0 =$

(i) $(2 \div 2) + 1 =$ ___ $+ 1 =$

(j) $(0 \times 2) \div (3 \times 1) =$ ___ $\div$ ___ $=$

Summary - Properties of Zero and One

$a + 0 = a$	$a \times 0 = 0$	$a \times 1 = a$
$a - 0 = a$	$0 \div a = 0$	$a \div 1 = a$
$a - a = 0$	$a \div 0$ is undefined	$a \div a = 1$

EXERCISE - Properties of Zero and One

1. Fill in the blanks.

(a) Since $7 + 0 = 7$, zero is called the ______________ element for ______________ or the ______________ identity.

(b) When zero is divided by a non-zero number, the result is ________.

(c) If the product of two whole numbers is zero, then at least one of the factors must be ________________.

(d) When any whole number is divided by zero, the result is ________________.

(e) When two identical natural numbers are divided, the result is ________________.

(f) Since $7 \times 1 = 7$, 1 is called the ______________ element for ________________ or the ________________ identity.

(g) Whenever zero appears as a factor, the product equals __________.

(h) A number remains unchanged when it is multiplied by __________. or when ________________ is added to it.

(i) Subtracting ______________ from a number leaves the number unchanged.

(j) A quotient of ______________ is obtained when the divisor and dividend are the same natural number.

(k) The result of subtracting two ______________ whole numbers is zero.

2. Tell whether each of the following statements is true or false. Rewrite each false statement in order to make it true.

	Statement	True or False?	Correction (if necessary)
(a)	$15 \times 1 = 1$	false	$15 \times 1 = 15$
(b)	$8 + 1 = 9$	true	no correction
(c)	$7 \times 0 = 7$		
(d)	$0 + 0 = 0$		
(e)	$5 + 0 = 0$		
(f)	$8 \div 8 = 0$		
(g)	$0 \div 5$ is undefined.		
(h)	$1 \times 0 = 0$		
(i)	$3 - 0 = 0$		
(j)	$12 \div 0$ is undefined.		
(k)	$14 \div 1 = 1$		

3. Simplify each of the following expressions. If the simplification is undefined write "undefined" beside the expression.

(a) $16 - 0 =$ ____

(b) $4 \times 1 =$ ____

(c) $0 \div 17 =$ ____

(d) $2 \times 0 =$ ____

(e) $14 - 14 =$ ____

(f) $12 \div 0 =$ ____

(g) $45 + 0 =$ ____

(h) $5 \div 5 =$ ____

(i) $31 \div 1 =$ ____

(j) $8 \times 7 \times 0 \times 4 \times 1 =$ ____

(k) $(4 \times 1) + (5 \times 0) =$ ___ + ___ = ____

(l) $(0 \div 9) + 3 =$ ___ + 3 = ____

(m) $4 \times (2 \times 0) = 4 \times$ ___ = ____

(n) $(6 \times 1) - (4 + 0) =$ ___ - ___ = ____

(o) $14(0 \div 6) = 14 \times$ ___ = ___

(p) $3(7 \div 0) =$ undefined

(q) $(3 - 3) \div 3 =$

(r) $1 \times (8 + 0) =$

(s) $(6 \div 6) \div 0 =$

(t) $(2 \times 0 \times 5) \div 10 =$

(u) $(2 \times 1 \times 5) \div 10 =$

(v) $9 \div (1 \times 9) =$

(w) $(6 + 0) \div (3 - 0) =$

(x) $(10 \div 0) + 1 =$

(y) $(4 \div 1) \div 0 =$

(z) $(4 \div 1) \div 1 =$

Key to Self-Correcting Exercises in Lesson 2

Exercise #1, page 4

1. (a) comm. prop. of add. Addends 12 and 13 have been switched.
   (b) comm. prop. of mult. Factors 7 and 3 have been switched.
   (c) comm. prop. of add. Addends 6 and 9 have been switched.
   (d) comm. prop. of mult. Factors 7 and $(6+9)$ have been switched.
   (e) comm. prop. of add. Addends (8×2) and 3 have been switched.
   (f) comm. prop. of mult. Factors 8 and 2 have been switched.
   (g) comm. prop. of mult. Factors a and b have been switched.
   (h) comm. prop. of add. Addends ab and c have been switched.

Exercise #2, page 10

1. (a) comm. prop. of add. Addends $2a$ and $3b$ have been switched.
   (b) assoc. prop. of mult. Factors appear in same position but are grouped differently.
   (c) comm. prop. of mult. Factors $(x+y)$ and z have been interchanged.
   (d) comm. prop. of add. Addends m and $3n$ have been interchanged.
   (e) comm. prop. of mult. Factors t and s have been switched.
   (f) assoc. prop. of mult. Grouping has been changed. (Order r, t, s is maintained.)
   (g) comm. prop. of mult. Factors $(m+n)$ and $(x+y)$ have been switched.
   (h) comm. prop. of add. Addends m and n have been switched.
   (i) comm. prop. of mult. Factors t and r were switched and factors m and s were switched.
   (j) assoc. prop. of add. Addends appear in same order but are grouped differently.
   (k) comm. prop. of add. Addends $3(m-n)$ and 7 have been switched.

Exercise #3, page 24

1. (a) Subtraction is not associative. (Grouping has been changed.)
   (b) Subtraction is not commutative. (Order has been changed.)
   (c) W is not closed under subtraction. ($3 - 6$ does not belong to W.)
   (d) Subtraction is not commutative. (Order of subtracting x and y has been changed.)
   (e) Subtraction is not associative. (Grouping has been changed.)
   (f) Subtraction is not commutative. (Order of subtracting $a - b$ and c has been changed.)

2. (a) Addition is commutative. (Order of adding x and 3 has been changed.)
   (b) Addition is associative. (No change in order, but grouping is different.)
   (c) W is closed under addition. ($10 + 11$ belongs to W.)
   (d) Multiplication is commutative. (Factors a and b-c have been switched.)
   (e) Addition is commutative. (Addends x and $y - z$ have been switched.)
   (f) W is closed under multiplication. (6×3 belongs to W.)
   (g) Multiplication is associative. (The 3 factors appear in the same order, but are grouped differently.)

Exercise #4, page 29

1. (a) Division is not associative. (Grouping has been changed.)
   (b) W is not closed under division. ($8 \div 6$ does not belong to W.)
   (c) Division is not commutative. (Order of dividing 6×2 and 4 has been changed.)
   (d) Division is not commutative. (Order of dividing has been changed.)
   (e) Division is not commutative. (Order of dividing b and c has been changed.)
   (f) Division is not associative. (Grouping has been changed.)

Exercise #5, page 38

1.	(a)	$3 \div 0$ is undefined	($a \div 0$ is undefined)
	(b)	$0 \div 3 =$ zero	($0 \div a = 0$)
	(c)	$0 \times 5 =$ zero	($0 \times a = 0$)
	(d)	$6 \times 0 \times 3 =$ zero	($0 \times a = 0$)
	(e)	$(8 - 8) \div 2 = 0 \div 2 =$ zero	($a - a = 0$ and $0 \div a = 0$)
	(f)	$5(9 - 9) = 5 \times 0 =$ zero	($a - a = 0$ and $a \times 0 = 0$)
	(g)	$4 \div (3 \times 0) = 4 \div 0 =$ undefined	($a \times 0 = 0$ and $a \div 0$ is undefined.)
	(h)	$(3 + 6) \div 0 = 9 \div 0 =$ undefined	($a \div 0$ is undefined.)
	(i)	$0 \div (3 + 0) = 0 \div 3 =$ zero	($a + 0 = a$ and $0 \div a = 0$)
	(j)	$4 \times 2 \times 0 =$ zero	($a \times 0 = 0$)

Exercise #5, cont'd

2. (a) $0 \times 3 = \underline{0}$ (Multiplying by zero gives a product of zero.)

   (b) $5 + 0 = \underline{5}$ (Adding zero leaves a number unchanged.)

   (c) $8 - 0 = 8$ (Subtracting zero leaves a number unchanged.)

   (d) $12 \div 0$ is undefined. (Quotient is undefined when divisor is zero.)

   (e) $0 \div 7 = \underline{0}$ (Quotient is zero when dividend is zero.)

   (f) $12 \div (6 - 6) = 12 \div 0$ is undefined.

   (g) $(2 \times 0) + 0 = 0 + 0 = \underline{0}$

   (h) $0 \times 9 \times 10 = \underline{0}$

   (i) $(9 + 0)2 = 9 \times 2 = \underline{18}$

   (j) $(0 \div 4)2 = 0 \times 2 = \underline{0}$

   (k) $0(1 + 2 + 3) = 0 \times 6 = \underline{0}$

   (l) $(0 \times 5) \div 16 = 0 \div 16 = \underline{0}$

   (m) $0 \div (3 - 0) = 0 \div 3 = \underline{0}$

   (n) $15 \div (3 + 0) = 15 \div 3 = \underline{5}$

Exercise #6, page 40

1. (a) $8 \div 8 = \underline{1}$ (Quotient of two identical numbers is 1.)

   (b) $1 \times 64 = \underline{64}$ (Multiplying by one leaves a number unchanged.)

   (c) $8 \div 1 = \underline{8}$ (Dividing by one leaves a number unchanged.)

   (d) $1 \times 0 = \underline{0}$ (Product is 0 since 0 is a factor.)

   (e) $0 + (4 \div 1) = 0 + 4 = \underline{4}$

   (f) $1 \times (3 \div 3) = 1 \times 1 = \underline{1}$

   (g) $(5 \div 5) \div 0 = 1 \div 0$ is undefined

   (h) $(6 \div 6) + 0 = 1 + 0 = \underline{1}$

   (i) $(2 \div 2) + 1 = 1 + 1 = \underline{2}$

   (j) $(0 \times 2) \div (3 \times 1) = 0 \div 3 = \underline{0}$

End of Lesson 2

Lesson 3

Integers

Basic Algebra and Geometry

INTEGERS

In this lesson, you are shown how to compute with integers (numbers having + or - signs). Future lessons in this course rely very heavily upon your ability to operate with these numbers. Try to master this topic right now so that you have a good background for algebra later.

Topic One: The Set of Integers

In Lesson 2 you learned that the set of whole numbers

$$W = \{0, 1, 2, \ldots\}$$

is not closed under subtraction. This means that the difference of two whole numbers is not always a whole number. For any two whole numbers a and b, the difference (a - b) represents a whole number only if a > b.

EXAMPLES:

1. (7 - 2) belongs to W since 7 is greater than 2 (7 > 2).
2. (2 - 7) does not belong to W since 2 is not greater than 7 ($2 \not> 7$).

Beside each difference below, write W if it represents a whole number.

1. 16 - 12	W	4. 0 - 12	______
2. 1 - 2	______	5. 14 - 13	______
3. 8 - 0	______	6. 70 - 90	______

The SET OF INTEGERS was invented in order to obtain closure under subtraction. This new set also helps us talk about temperatures below zero, debts, losses, and anything involving the idea of "less than zero."

A. Definition of Integers

The set of integers contains elements which are less than zero besides the elements which belong to W, the set of whole numbers.

The integers which are less than zero are called NEGATIVE INTEGERS and are obtained by placing a minus sign in front of each natural number. Thus, the negative integers are represented by:

$$-1,\ -2,\ -3,\ -4,\ -5,\ -6,\ \ldots$$

The familiar natural numbers can be called POSITIVE INTEGERS and may be written with plus signs in front of them as follows:

$$+1,\ +2,\ +3,\ +4,\ +5,\ +6,\ \ldots$$

The set of integers is composed of:

1. the positive integers
2. the negative integers
3. the whole number zero (which is neither positive nor negative).

We use the following notation to represent the infinite set of integers:

$$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Note that, for convenience, we will always designate this set by the capital letter I.

Does -6 belong to set I? ________ Does 6? ________ Does $\frac{1}{2}$? no

Does 3.2? ________ Does 0? ________ Does -100? ________

Does $1\frac{2}{3}$? ________

B. Using Integers

Positive and negative integers are useful in helping us distinguish between temperatures that are above zero and those that are below zero. We label those readings that are above zero as positive (+) and those that are below zero as negative (-). For example, a temperature of 30° above zero could be represented by +30 while a temperature of 30° below zero could be represented by -30. How could you represent 75° above zero? ________ 10° below zero? ________

Positive and negative integers are also useful in distinguishing between profits and losses. For example, a profit of $20 could be represented by +20 while a loss of $20 could be represented by -20. How could you represent a loss of $100? ________

a profit of $9? ________.

Use a positive or negative integer to represent each of the following situations. (Don't include units or dollar signs.)

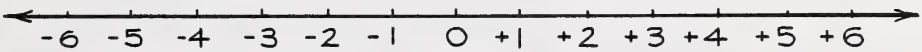
1. a loss of 20 yards ________
2. a gain of 5 yards ________
3. an increase of $30 +30
4. a debt of $500 ________
5. a balance of $150 ________
6. 1800 ft. below sea level ________
7. a deduction of 5 marks ________
8. $1500 in assets ________
9. a deficit of $110 ________
10. 100 ft. above sea level ________

C. The Integer Number Line

The set of integers may be associated with points on a straight line. This can be done by extending the whole number line to the left. The same scale that was used to locate the whole numbers must be used to locate the negative integers to the left of zero. The point which is assigned the number zero is called the ORIGIN. For every point to the right of the origin that is labeled a, there is a point an equal distance to the left of the origin that is labeled -a. The process of marking off equal segments on each side of the origin continues indefinitely, even though only a limited number of points can be shown on the line.

Note that arrowheads occur at both ends of the integer number line. This indicates that the line goes on indefinitely in <u>both</u> directions. Every integer can be represented by a point on the line.

The positive integers can be placed on the number line with plus signs in front of them to emphasize their direction with respect to the origin.



The + and - signs used in this manner are called SIGNS OF DIRECTION since they indicate whether a point lies to the right or left of the origin. Any positive integer lies to the right of the origin and any negative integer lies to the left of the origin.

State whether the point corresponding to each of the following integers lies to the left or right of the origin.

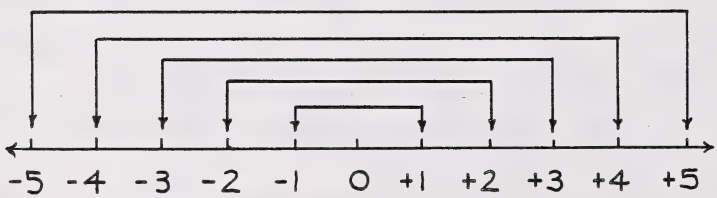
1. -2 left of the origin
2. -17 ______
3. 6 ______
4. +3 ______
5. -1 ______
6. 1 ______

We can think of all the integers, except zero, as possessing both magnitude and direction. The distance between a number and the origin on the number line is called the MAGNITUDE of that number. The sign in front of the number tells us the direction of the number with respect to the origin. For example, the integer +2 has a magnitude of 2 since it is a distance of 2 units from the origin. It has a positive direction and lies to the right of the origin. The integer -7 has a magnitude of 7 since it is a distance of 7 units from the origin. It has a negative direction and lies to the left of the origin.

State the magnitude and direction of each of the following integers.

	Magnitude	Direction
1. $+(4 \times 3)$	12	right of origin
2. -4	______	______
3. 100	______	______
4. $-(8 \div 4)$	______	______
5. $(5 + 2)$	______	______
6. $-a$ if $a \in N$	______	______

Two integers which have the same magnitude and opposite direction are called OPPOSITES. Every integer, except zero, has as its opposite a number on the other side of the origin at an equal distance from the origin.



For example, the integers -1 and +1 both have magnitudes of 1, but have opposite direction. Thus, they are called opposites.

Similarly,

-2 and ______ are opposites.

+3 and ______ are opposites.

12 and ______ are opposites.

The magnitude of an integer is also referred to as being the ABSOLUTE VALUE of that integer. The absolute value of an integer is indicated by placing the integer between vertical bars.

EXAMPLES:

$|+6| = 6$ This is read, "the absolute value of positive 6 is 6."

$|-6| = 6$ This is read, "the absolute value of negative 6 is 6."

$|0| = 0$ This is read, "the absolute value of 0 is 0."

Note that the absolute value of opposite integers are equal. i.e. $|+6| = |-6| = 6$.

ABSOLUTE VALUE
For any <u>whole number</u> "a",
$|+a| = a$
$|-a| = a$
EXAMPLES: $|+7| = 7$, $|-7| = 7$,
$|+2| = 2$, $|-2| = 2$

Thus,

1. The absolute value of -3 is written $|-3|$ and equals _____.
2. The absolute value of +5 is written _____ and equals _____.
3. The absolute value of -12 is written _____ and equals _____.

What two integers both have an absolute value of 8? _____, _____.

Self-correcting Exercise #1

Answers may be found on page 47 of this lesson.

1. Simplify the following expressions.

(a) $|-7| \times |-3| = \_\_\_\_ \times \_\_\_\_ = \_\_\_\_$

(b) $|8 - 5| = |\_\_| = \_\_\_\_$

(c) $|-10| + |5| = \_\_\_\_ + \_\_\_\_ = \_\_\_\_$

(d) $|-16| \div |4| = \_\_\_\_ \div \_\_\_\_ = \_\_\_\_$

2. State which of the following absolute values is greater.

		Greater
(a) $\lvert -2 \rvert$	$\lvert -14 \rvert$	$\lvert -14 \rvert$
(b) $\lvert -7 \rvert$	$\lvert 2 \rvert$	______
(c) $\lvert 13 \rvert$	$\lvert 12 \rvert$	______
(d) $\lvert 5 \rvert$	$\lvert -6 \rvert$	______
(e) $\lvert 0 \rvert$	$\lvert -2 \rvert$	______
(f) $\lvert 4 \rvert$	$\lvert -5 \rvert$	______

D. <u>Ordering of the Integers</u>

The numbers are ordered on the integer line in the same way they are ordered on the whole number line. For any integers a and b, if point a lies to the <u>right</u> of point b on the number line, we can say that a is <u>greater</u> than b and can write $a > b$.

EXAMPLES:
Since 4 is to the right of 2, $4 > 2$.
Since 2 is to the right of -1, $2 > -1$.
Since 2 is to the ______ of -4, 2 ___ -4.
Since -3 is to the ______ of -6, -3 ___ -6.
} Fill in the blanks.

If any point "a" lies to the <u>left</u> of point b on the integer line we can say that a is <u>less</u> than b and can write $a < b$.

EXAMPLES:
Since 3 is to the left of 5, $3 < 5$.
Since -1 is to the left of 1, $-1 < 1$.
Since -3 is to the ______ of -2, -3 ___ -2.
Since -4 is to the ______ of 2, -4 ___ 2.
} Fill in the blanks.

Whenever you are asked to compare two integers, think of their positions with respect to each other on the number line. The integer which appears on the number line in the <u>farther left</u> position will be the <u>smaller</u> of the two integers. The integer which appears in the <u>farther right</u> position will be the <u>larger</u> of the two integers.

Self-correcting Exercise #2

Answers may be found on page 48 of this lesson.

1. Decide whether each of the following statements is true or false. (Think of how the two numbers lie in relation to each other on the number line.

		True or False?
(a)	$-2 < -6$	____________
(b)	$-5 < -4$	____________
(c)	$-4 < 2$	____________
(d)	$0 > 3$	____________
(e)	$6 > -2$	____________
(f)	$12 > -14$	____________
(g)	$-8 < 0$	____________
(h)	$-12 > -6$	____________

2. Between each pair of integers, insert the symbol > or < to make the statement true.

(a) -1 _____ -8 (b) 7 _____ 3

(c) -7 _____ -3 (d) -5 _____ 7

(e) 0 _____ -3 (f) 8 _____ -12

E. Specifying Subsets of the Set of Integers

On pages 24-26 of Lesson 1, you had some practice specifying subsets of W (the set of whole numbers) by tabulation and by using set-builder notation. Now let us try specifying some subsets of I.

EXAMPLE 1: Specify the set of all integers less than -2.
Set-builder notation: $\{x \mid x < -2, x \in I\}$. This is read "the set of all x such that x is less than -2 and x is an integer." Since this set contains all the integers less than -2, all the integers that are to the left of -2 on the I-line are included in this set.
Tabulation: $\{-3, -4, -5, \ldots\}$

Does $-5\frac{1}{2}$ belong to this set? no Does -6? _____ Does 0? _____

Does 1? _____ Does -1? _____ Does 8? _____ Does -8? _____

EXAMPLE 2: Specify the set of all integers that are less than 3 and greater than or equal to -7.
Set-builder notation: $\{x \mid -7 \leqq x < 3, x \epsilon I\}$. Note that the arrows point to the left. Since -7 is smaller than 3, it is placed on the left and 3 is placed on the right. The integer -7, and all the integers that fall between -7 and 3 on the number line will belong to this set.
Tabulation: $\{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2\}$

Does $-5\frac{1}{2}$ belong to this set? _____ Does -6? _____ Does 0? _____

Does 1? _____ Does -1? _____ Does 8? _____ Does -8? _____

Self-correcting Exercise #3

Answers may be found on page 48 of this lesson.

1. Write the following statements in symbols.

   (a) x is between -7 and -2 inclusive. $-7 \leqq x \leqq -2$

   (b) x is less than -5. _____

   (c) x is greater than or equal to -10. _____

   (d) x is between 7 and -3. _____

   (e) x is less than or equal to -50 and greater than -100. _____

   (f) x is greater than -4 and less than 3. _____

2. Specify the following sets using set-builder notation. Then tabulate each set.

   (a) The set of all integers greater than or equal to -5.

   Set-builder notation: _____

   Tabulation: _____

   (b) The set of all integers between -100 and 50 inclusive.

   Set-builder notation: _____

   Tabulation: _____

(c) The set of all integers greater than -13 and less than or equal to -6.

Set-builder notation: ____________________

Tabulation: ____________________

F. Graphing Subsets of I

The graph of a given subset of the set of integers is the set of points on the integer number line whose coordinates are members of the given set of numbers. To graph a subset of I, we make a heavy dot on the number line at each point corresponding to an element of the subset.

EXAMPLE 1: Graph the set $A = \{-6, -3, 0, 2, 4\}$.

Solution

We must place dots on the number line at the points corresponding to the integers -6, -3, 0, 2, and 4.

How many points are there in the graph of set A? __________

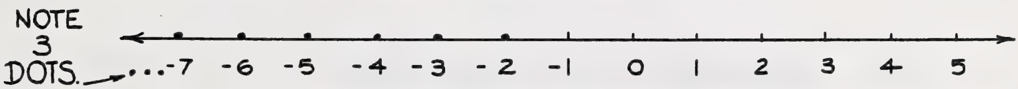
EXAMPLE 2: Graph the set $B = \{x \mid x \leq -2, x \in I\}$

Solution

Since x must be less than or equal to -2, we are interested in the point that represents -2 and all the points that lie to the left of -2 on the number line. Set B may be tabulated as follows:

$$B = \{-2, -3, -4, \ldots\}$$

Since we cannot graph all the elements of set B, we can indicate that the graph goes on indefinitely to the left by placing 3 dots to the left of the smallest integer given on the number line.



Are there an infinite number of points in the graph of set B? __________

Does -1 belong to set B? _____ Does -12? _____ Does 5? _____

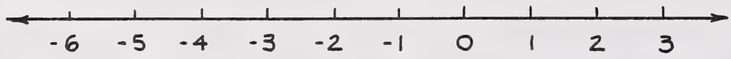
Does 0? _____ Does -2? _____ Does -8? _____ Does -20? _____

Self-correcting Exercise #4

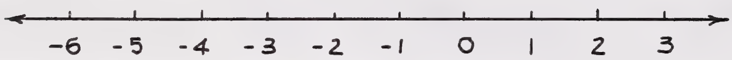
Answers may be found on page 48 of this lesson.

1. Graph each of the following sets on the number line provided.

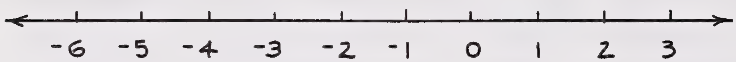
(a) $A = \{3, 1, -2, -1, -5, 2\}$



(b) $B = \{x \mid x > -3,\ x \in I\}$
First, tabulate set B. $B = \{$ ______________________ $\}$



(c) $C = \{y \mid -6 < y \leqq -1,\ y \in I\}$
First, tabulate set C. $C = \{$ ______________________ $\}$.



EXERCISE - The Set of Integers

1. Fill in the blanks.

(a) Numbers with plus sign designate points to the ____________________ of the origin.

(b) The integer ____________ is neither positive nor negative.

(c) The number corresponding to the point 4 units to the left of the origin is written _______________.

(d) A number on the integer line is ________________________ than any number to the left of it.

(e) Zero is less than any number to the ______________ of it on the number line.

(f) Stating that $n < 0$ where $n \in I$, is equivalent to saying that n is a ______________ integer.

(g) The distance between a number and the origin is called the ______________ of the number.

(h) The absolute value of the integer $(a + b)$ is written ______________.

(i) The integers -5 and +5 have the same ______________ but opposite ______________.

(j) On the number line, the point corresponding to the integer -6 lies one unit to the left of the point corresponding to ______________.

(k) All positive integers are ______________ than zero.

(l) The opposite of a negative integer is always a ______________ integer.

(m) The set of ______________ numbers is the same as the set of positive integers.

2. Arrange each list of numbers in order. Begin with the least number.

(a) 3, -4, 1, -1 -4, -1, 1, 3.

(b) 2, -6, -4, -7, 0, 5 ______________

(c) 10, -10, 5, -5, 0, -20, 20 ______________

(d) 2, -4, 6, -8, 10, -12, 14, -16 ______________

(e) -1, 6, -8, -5, 0, 1, -3 ______________

3. Tell whether each of the following statements is true or false. (Review page 5.)

True or False?

(a) $|9| = 9$ ________

(b) $|-345| = -345$ ________

(c) $|33| = |-33|$ ________

(d) $|-8| < 0$ ________

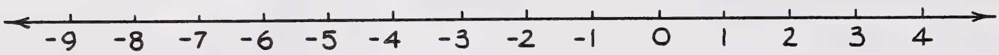
(e) $|6| > 0$ ________

(f) $|32| > |-56|$ ________

(g) $|-27| < |19|$ ________

(h) $|18| < |-20|$ ________

4.



(a) What is the largest integer shown on this number line? 4

(b) What is the smallest integer shown on this number line? ________

(c) Name three integers on the line that are less than -6.

____

(d) Name three integers that are between -3 and 1. ________

(e) Name three integers between -5 and -3 inclusive.

____

(f) Name a negative integer that is greater than -2. ________

(g) Name three integers that are less than -4 but greater than or equal to -8. ________

5. Tabulate the following sets.

(a) the set of positive integers

$\{1, 2, 3, \ldots\}$

(b) the set of negative integers

____

(c) the set of integers

____

(d) the set of non-negative integers (i.e. the set of positive integers and zero)

____

(e) the set of non-positive integers

____

(f) the set of all integers greater than or equal to -1.

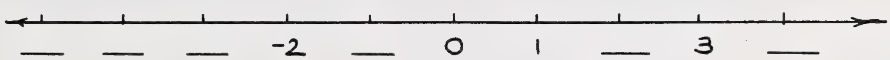
____

(g) the set of all integers greater than or equal to -3 and less than 3.

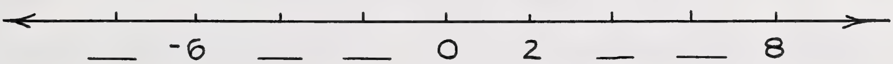
____

6. Insert each of the missing integers under the number lines.

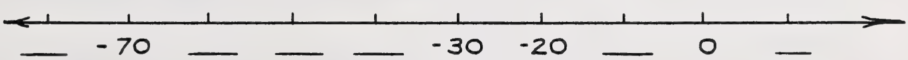
(a)



(b)



(c)



7. Indicate the relation between each of the following pairs of numbers by inserting the symbol < or >. Then state whether the first number is to the right or left of the second number on the number line..

EXAMPLE: $-(5+2) > -(6 \times 5)$ since -7 is to the right of -30.

(a) 13 _____ 17 since ______________________________

(b) 10 _____ 0 since ______________________________

(c) 10 _____ -10 since ______________________________

(d) -10 _____ 0 since ______________________________

(e) -10 _____ -20 since ______________________________

(f) -13 _____ -12 since ______________________________

(g) $-(5 \times 1)$ _____ (3×0) since ______________________________

(h) $-(3 + 0)$ _____ $-(6 - 0)$ since ______________________________

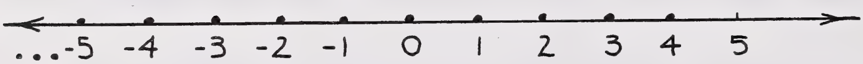
8. Specify each set by using set-builder notation. Then tabulate the set and draw its graph.

(a) The set of all integers less than or equal to 4.

Set-builder notation: $\{x \mid x \leqq 4, x \in I\}$

Tabulation: $\{4, 3, 2, 1, 0, -1, -2, \ldots\}$

Graph:



(b) The set of all integers between 4 and -4.

Set-builder notation: ______________________

Tabulation: ______________________

Graph:

(c) The set of all integers greater than -4 and less than or equal to 2.

Set-builder notation: ____________________

Tabulation: ____________________

Graph:

(d) The set of all integers greater than -3.

Set-builder notation: ____________________

Tabulation: ____________________

Graph:

Topic Two: Adding Integers

A. Trips on the Number Line

If we learn how to represent integers as trips on a number line, we can use this knowledge to help us add integers.

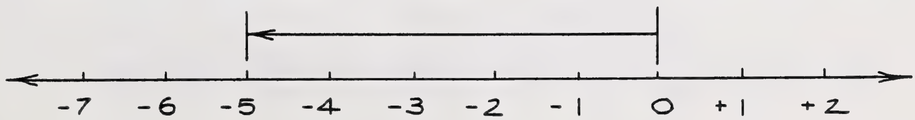
Trips to the right on the number line can be used to represent positive integers and trips to the left to represent negative integers. The magnitude (or absolute value) of an integer affects the length of the trip. A trip can be indicated on the number line by drawing a ray above the number line that begins at zero and ends at the integer in question. An arrow at the end of the ray tells us whether the trip has been to the left or to the right. Vertical bars placed at both ends of the ray mark the starting point and end point of the trip.

EXAMPLE: Show a trip of -5 on the number line.

Solution

Since -5 is a negative number, the trip is to the left. Draw a ray that begins at 0 and ends at -5. (Put an arrowhead at the end of the ray.)

Draw vertical bars above -5 and 0.

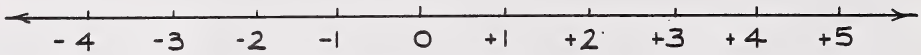


Self-correcting Exercise #5

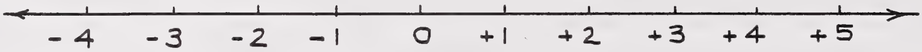
Answers may be found on page 49 of this lesson.

1. Represent each integer as a trip on the number line.

(a) 4

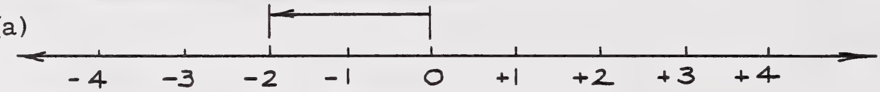


(b) -3



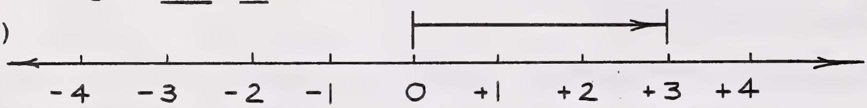
2. Name the integer that corresponds to each of the following trips.

(a)



Integer: ___ _

(b)



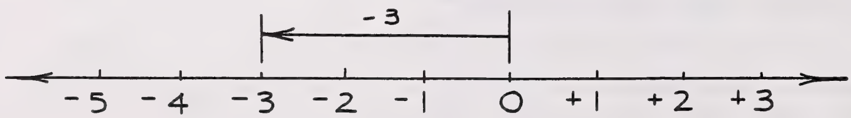
Integer: ___

If we are going to use the number line to find the sum of two integers, we need to know how to represent two consecutive integer trips on the number line. We can do this by representing the first integer by a trip starting at zero. Then, we can represent the second integer by a trip starting where the first trip left off. In both cases, the sign of the integer affects the direction of the trip and the magnitude of the integer affects the length of the trip. The integer that lies below the end of the second trip should be circled since it represents the sum of the two integers.

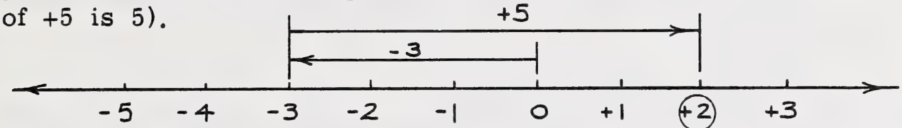
EXAMPLE: Represent the sum (-3) + (+5) on the number line. Then, use your diagram to determine the value of the sum.

Solution

The integer -3 can be represented on the number line by drawing a ray that begins at 0 and ends at -3.



Then, beginning at -3, we must show a trip of +5. We can do this by drawing a ray that begins at -3, goes to the right (since +5 is positive) and covers 5 spaces on the number line (since the magnitude of +5 is 5).



Note that the second ray ends at +2. This number is circled since it represents the sum of -3 and +5.

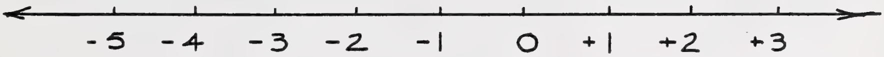
(-3) + (+5) = +2

<u>Self-correcting Exercise #6</u>

Answers may be found on page 49 of this lesson.

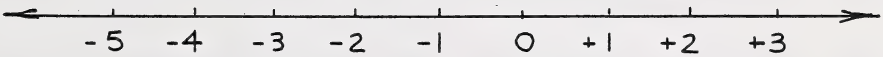
1. Represent each sum on the number line provided. Then use your diagram to determine the value of the sum.

(a) (+2) + (-6)



(+2) + (-6) = ______

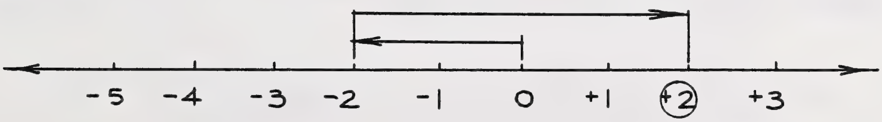
(b) (-1) + (-3)



(-1) + (-3) = ______

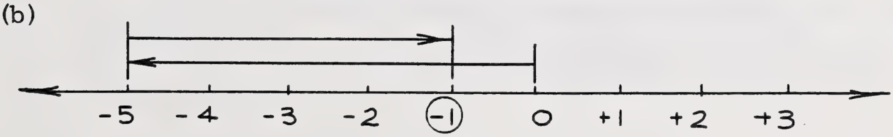
2. Write an addition sentence for each number line picture.

(a)



(___) + (___) = ______

(b)



(___) + (___) = ______

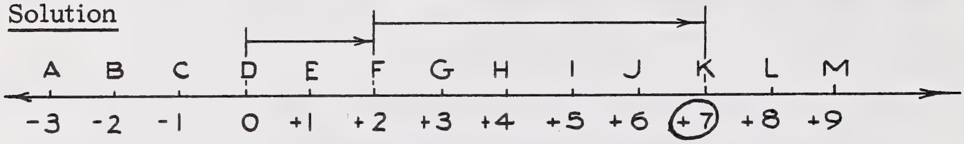
B. Rules for Adding Integers

Now, let us use our number line pictures to arrive at some rules for adding integers.

1. Addition of two positive integers

EXAMPLE: (+2) + (+5) =

Solution



The integer +2 represents a trip of 2 units to the ______________ of the origin. On the number line above, the integer +2 is represented by a trip from point _____ to point F. From point F, you must move ________ units to the ______________ to represent the integer +5. In the second trip, you go from point F to point _____. Thus, the sum (+2) + (+5) could also be represented by a single trip of +7.

(+2) + (+5) = +7

On a piece of scrap paper, use number lines to help you find the following sums.

(+4) + (+2) = _____	(+7) + (+9) = _____
(+2) + (+4) = _____	(+15) + (+3) = _____

Look at your four answers above. Note that in every case the sum is a positive integer. Now, examine the absolute value of each sum. Note that it can be obtained by finding the sum of the absolute values of the two numbers.

> **ADDITION-TWO POSITIVE INTEGERS**
>
> To add two positive integers, add their absolute values and give the result a positive sign.

THE SUM IS POSITIVE SINCE BOTH NUMBERS ARE POSITIVE.

EXAMPLE:

$$\left.\begin{aligned}(+4) + (+8) &= +[\,|+4| + |+8|\,] \\ &= +[\;4 + 8\;]\end{aligned}\right\}$$

Think these two steps.
Don't write them down.

$$= +12$$

FIND THE SUM OF THE ABSOLUTE VALUES.

Using the rule for adding two positive integers, find the following sums.

(a) (+3) + (+1) = +4

(b) (+9) + (+3) = ____

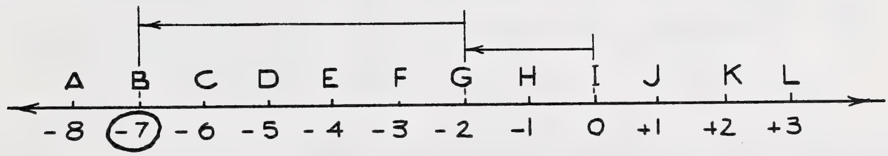
(c) (+6) + (+3) = ____

(d) (+100) + (+5) = ____

2. Addition of two negative integers

EXAMPLE: (-2) + (-5) =

Solution



The integer -2 represents a trip of ________ units to the ________ of the origin. On the number line above, you move from point _____ to point _____. Then, the integer -5 represents a trip of _____ units to the ________.

On the number line, you move from point G to point _____. Thus, the sum (-2) + (-5) could also be represented by a single trip of -7.

$$(-2) + (-5) = -7$$

On a piece of scrap paper, use number lines to help you find the following sums.

(-4) + (-2) = ____

(-2) + (-4) = ____

(-7) + (-9) = ____

(-15) + (-3) = ____

Look at your four answers above. Note that in every case the sum is a negative integer. Now, examine the absolute value of each sum. Note that it can be obtained by finding the sum of the absolute values of the two numbers.

ADDITION-TWO NEGATIVE INTEGERS

To add two negative integers, add their absolute values and give the result a negative sign.

EXAMPLE:

THE SUM IS NEGATIVE SINCE BOTH NUMBERS ARE NEGATIVE.

FIND THE SUM OF THE ABSOLUTE VALUES.

$$\left.\begin{aligned} (-4) + (-8) &= -[\,|-4| + |-8|\,] \\ &= -[\,4 + 8\,] \\ &= -12 \end{aligned}\right\}$$

Think these two steps. Don't write them down.

Using the rule for adding two negative integers, find the following sums.

(a) $(-3) + (-1) =$ -4

(b) $(-9) + (-3) =$ _____

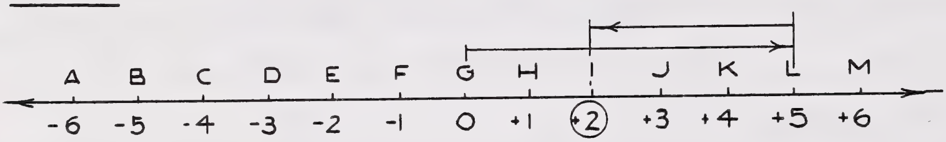
(c) $(-6) + (-3) =$ _____

(d) $(-100) + (-5) =$ _____

3. Addition of two integers with unlike signs

EXAMPLE: $(+5) + (-3) =$ ▨

Solution



The integer +5 represents a trip of _____ units to the _____.

On the number line above, you move from point _____ to point _____.

From this point, you go _____ units to the _____ to represent the integer -3. In the second trip you go from point L to point _____.

Thus, a trip of +5 followed by a trip of -3 is equivalent to a single trip of +2.

$$(+5) + (-3) = +2$$

On a piece of scrap paper, use number lines to help you find the following sums.

(+8) + (-5) = _____ (+3) + (-7) = _____

(-8) + (+5) = _____ (-3) + (+7) = _____

Look at your four answers above. Note that in every case the sign of the sum corresponds to the sign of the number with the larger absolute value. Now, examine the absolute value of each sum. Note that it can be obtained by subtracting the absolute values of the two numbers. (Subtract the smaller absolute value from the larger.)

> ADDITION-TWO INTEGERS WITH UNLIKE SIGNS
>
> To add two integers with unlike signs, find the difference of their absolute values, and give the result the sign of the number with the larger absolute value.

EXAMPLES:

SINCE THE POSITIVE INTEGER HAS THE LARGER ABSOLUTE VALUE, THE SUM IS POSITIVE.

$(+11) + (-5) = +[\,|+11| - |-5|\,]$
$= +[\,11 - 5\,]$

Think these two steps. Don't write them down.

$= +6$

FIND THE DIFFERENCE OF THE ABSOLUTE VALUES.

SINCE THE NEGATIVE INTEGER HAS THE LARGER ABSOLUTE VALUE, THE SUM IS NEGATIVE.

$(+2) + (-7) = -[\,|-7| - |+2|\,]$
$= -[\,7 - 2\,]$
$= -5$

FIND THE DIFFERENCE OF THE ABSOLUTE VALUES.

$(+3) + (-3) = |+3| - |-3|$
$= 3 - 3$
$= 0$

Since the absolute values are the same, the sum is zero.

Using the rule for adding two integers with unlike signs, find the following sums.

(a) $(+8) + (-12) =$ −4

(b) $(+8) + (-2) =$ ____

(c) $(-9) + (+1) =$ ____

(d) $(-9) + (+10) =$ ____

Self-correcting Exercise #7

Answers may be found on page 50 of this lesson.

1. Add. Use the rules for finding the sum of two positive or two negative integers.

(a) $(+15) + (+6) =$ ____

(b) $(+1) + (+99) =$ ____

(c) $(-3) + (-16) =$ ____

(d) $(+30) + (+20) =$ ____

(e) $(-12) + (-12) =$ ____

(f) $(-6) + (-4) =$ ____

2. Add. Use the rules for finding the sum of two integers with unlike signs.

(a) $(+12) + (-9) =$ ____

(b) $(+17) + (-25) =$ ____

(c) $(-35) + (+15) =$ ____

(d) $(-13) + (+19) =$ ____

(e) $(-16) + (+20) =$ ____

(f) $(-11) + (+11) =$ ____

3. First, decide if each question represents the sum of two positive integers, the sum of two negative integers, or the sum of two integers with unlike signs. Then, use the appropriate rule to find the sum.

(a) $(+5) + (-7) =$ ____

(b) $(+7) + (-10) =$ ____

(c) $(-4) + (+10) =$ ____

(d) $(-3) + (-8) =$ ____

(e) $(-6) + (+8) =$ ____

(f) $(+13) + (-4) =$ ____

(g) $(+4) + (+9) =$ ____

(h) $(-23) + (-14) =$ ____

(i) $(+8) + (-8) =$ ____

(j) $(+2) + (+15) =$ ____

4. Find each sum by adding the first two numbers and then adding the third number to the result.

(a) $[(+8) + (-2)] + (-9) = (\_\_\_\_) + (-9) = \_\_\_\_$

(b) $[(-3) + (-2)] + (+1) = (\_\_\_\_) + (+1) = \_\_\_\_$

(c) $[(+4) + (+7)] + (-3) = (\_\_\_\_) + (-3) = \_\_\_\_$

(d) $[(-7) + (+4)] + (-8) = (\_\_\_\_) + (-8) = \_\_\_\_$

(e) $[(-5) + (-2)] + (-6) = (\_\_\_\_) + (-6) = \_\_\_\_$

SUMMARY-ADDING INTEGERS

1. If the signs are alike, add the absolute values and attach the common sign.

   e.g. $(+7) + (+3) = +10$
   $(-7) + (-3) = -10$

2. If the signs are unlike, subtract the absolute values and attach the sign of the integer with the larger absolute value.

   e.g. $(+7) + (-3) = +4$
   $(-7) + (+3) = -4$

<u>EXERCISE-Adding Integers</u>

1. Fill in the blanks.

(a) The sum of two positive integers is always a ________ integer.

The absolute value of the sum is determined by ____________ the absolute values of the two numbers.

(b) The sum of two negative integers is always a ________ integer.

The absolute value of the sum is determined by ____________

____________________________.

(c) When you add a positive integer and a negative integer:

(i) the absolute value of the sum is the ____________ between the greater absolute value and the smaller.

(ii) the sum is positive if the greater absolute value belongs to the ____________ integer.

(iii) the sum is negative if the greater absolute value belongs to the ____________ integer.

2. Add. First decide if the signs are like or unlike and then use the appropriate rule.

(a) $(-15) + (+8) =$ _____

(b) $(+15) + (-8) =$ _____

(c) $(+15) + (+8) =$ _____

(d) $(-15) + (-8) =$ _____

(e) $(+15) + (-15) =$ _____

(f) $(-36) + (+41) =$ _____

(g) $(-25) + (+19) =$ _____

(h) $(+21) + (-9) =$ _____

(i) $(-8) + (-11) =$ _____

(j) $(+3) + (-2) =$ _____

(k) $(-9) + (+9) =$ _____

(l) $(-5) + (+4) =$ _____

(m) $[(-5) + (-4)] + (+3) =$ ___ $+ (+3) =$ _____

(n) $[(-8) + (+7)] + (-4) =$ ___ $+ (-4) =$ _____

(o) $(-3) + [(+7) + (+8)] = (-3) +$ ___ $=$ _____

(p) $(+6) + [(-1) + (-5)] = (+6) +$ ___ $=$ _____

(q) $[(+8) + (-8)] + (-7) =$

(r) $(+11) + [(-5) + (-7)] =$

Topic Three: Subtracting Integers

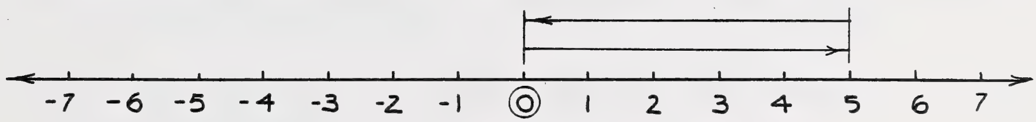
A. The Additive Inverse of an Integer

In our discussion of the integer number line, we found that every integer except zero has as its opposite an integer on the other side of the origin at an equal distance from the origin. For example, the integers +5 and -5 are OPPOSITES because they have the same magnitude (absolute value) but opposite direction.

What is the result when we add two opposites? Note that:

$$\begin{aligned}(+5) + (-5) &= |+5| - |-5| \\ &= 5 - 5 \\ &= 0\end{aligned}$$

This addition could be shown on the number line as follows:



A trip of 5 units to the right followed by a trip of 5 units to the left takes you back to your original starting position, the origin.

We shall now call the opposite of an integer its ADDITIVE INVERSE. One integer is the additive inverse of another integer if their sum is zero. Two integers which are additive inverses of each other have the same absolute value.

DEFINITION - ADDITIVE INVERSE OF AN INTEGER
For every integer a, there is a unique integer -a such that: $a + (-a) = (-a) + a = 0.$ "a" is the additive inverse of "-a" and "-a" is the additive inverse of "a".

EXAMPLES:

(+7) is the additive inverse of (-7) since $(+7) + (-7) = 0$.
(-3) is the additive inverse of (+3) since $(-3) + (+3) = 0$.
0 is the additive inverse of 0 since $0 + 0 = 0$.

Remember that the additive inverse of a positive number is always negative, the additive inverse of a negative number is always positive, and the additive inverse of zero is always zero.

What is the additive inverse of -4? _____ of 6? _____ of x?_____

Fill in the blanks below.

1. (-2) + (+2) = _____
2. _____ + (-3) = 0
3. (+7) + ____ = 0
4. (-14) + ____ = 0
5. (+6) + (-6) = _____
6. x + ____ = 0

B. <u>Rule for Subtracting Integers</u>

Review the definition of subtraction of whole numbers on page 20, Lesson 2. Subtraction in set I can be defined in a similar manner.

DEFINITION OF SUBTRACTION IN SET I
For any three integers a, b, and c, a - b = c if and only if a = b + c
"Subtract b from a" means "find the number c which must be added to b to make a."

EXAMPLE: 5 - (-10) = ▨

<u>Solution</u>. In the expression 5 - (-10), note that the first minus sign indicates the operation of subtraction. The second minus sign indicates a negative number. We could change this subtraction question to an addition question by asking ourselves, "What number must be added to -10 to make 5?"

i.e. (-10) + ▨ = 5

From our rules for adding two integers we know that:

$$(-10) + \boxed{+15} = 5.$$

Changing back to a subtraction question, we can say that:

$$5 - (-10) = \boxed{+15}$$

Note that we get the same result (+15), when we add 5 and the additive inverse of -10.

i.e. $5 + (+10) = \boxed{+15}$

Therefore, $\underbrace{5 - (-10)}_{\text{SUBTRACT } -10} = \underbrace{5 + (+10)}_{\text{ADD } +10} = \boxed{15}$

From the above example, we can see that subtraction in set I can be defined in terms of adding the additive inverse of the subtrahend. A general rule for subtracting two integers can be formulated as follows:

SUBTRACTION - TWO INTEGERS
For any two integers a and b, $a - b = a + (-b)$.

In general, any subtraction problem involving integers can be changed to an equivalent addition problem by making two adjustments:

1. Change the operation from subtraction to addition.
2. Give the additive inverse of the subtrahend (the number you are subtracting.)

Once you have changed the subtraction problem to an addition problem, you can find the sum by using the rules for adding integers.

EXAMPLES:

$(+3) - (+2) = (+3) + (-2) = +1$
$(+5) - (+6) = (+5) + (-6) = -1$
$(-1) - (-2) = \underbrace{(-1) + (+2)} = +1$

CHANGE OPERATION TO ADDITION AND GIVE ADDITIVE INVERSE OF <u>SECOND</u> NUMBER.

Self-correcting Exercise #8

Answers may be found on page 50 of this lesson.

1. Change each subtraction question to an addition question. This can be done by changing the operation to addition and giving the additive inverse of the second integer. (The first integer is left unchanged.)

(a) $(+8) - (+4) =$ ___ + ___

(b) $(-3) - (-3) =$ ___ + ___

(c) $(+4) - (+10) =$ ___ + ___

(d) $(+1) - (-4) =$ ___ + ___

(e) $(+2) - (+9) =$ ___ + ___

(f) $(-6) - (+3) =$ ___ + ___

2. Rewrite each subtraction question as an addition question. Then decide if the signs of the two integers in the sum are alike or different. In the last column, give the sum.

ADD ADDITIVE INVERSE OF -9

		Corresponding addition question	Are the signs alike or different?	Sum
(a)	(+13) - (-9)	(+13) + (+9)	________	________
(b)	(-6) - (+7)	(__) + (__)	________	________
(c)	(-3) - (-5)	(__) + (__)	________	________
(d)	(-4) - (+5)	(__) + (__)	________	________
(e)	(+23) - (+7)	(__) + (__)	________	________

EXERCISE - Subtracting Integers

1. Fill in the blanks.

(a) When the sum of two integers is ________, each is the opposite or ____________ ____________ of the other.

(b) The additive inverse of a positive integer is a ______________ integer having the __________ absolute value.

(c) Subtracting the integer b from the integer a is the same as ______________ the additive inverse of __________ to a.

(d) The subtraction question (-2) - (-5) can be changed to the addition question (___) + (___). Then, the sum can be found by using the rule for adding two integers with ____________ signs.

2. Convert each subtraction question to an addition question. Then use the rules for adding integers to find the sum.

ADDITIVE INVERSE OF −4

(a) $(+16) - (-4) = (\underline{+16}) + (\underline{+4}) = \underline{+20}$

(b) $(-18) - (+13) = (\_\_) + (\_\_) = \_\_\_$

(c) $(-6) - (-8) = (\_\_) + (\_\_) = \_\_\_$

(d) $(-9) - (-9) = (\_\_) + (\_\_) = \_\_\_$

(e) $(+6) - (+14) = (\_\_) + (\_\_) = \_\_\_$

(f) $(+12) - (+3) = (\_\_) + (\_\_) = \_\_\_$

ADDITIVE INVERSE OF −3

ADDITIVE INVERSE OF +4

(g) $[(-7) - (-3)] - (+4) = [(\underline{-7}) + (\underline{+3})] + (\underline{-4}) = (\underline{-4}) + (\underline{-4}) = \underline{-8}$

(h) $[(+4) - (+1)] - (-8) = [(\_\_) + (\_\_)] + (\_\_) = (\_\_) + (\_\_) = \_\_\_$

(i) $[(+6) - (-2)] - (+8) = [(\_\_) + (\_\_)] + (\_\_) = (\_\_) + (\_\_) = \_\_\_$

Topic Four: Multiplying and Dividing Integers

Multiplication in the set of whole numbers can be defined as repeated addition.

EXAMPLES:

3×4 means that 4 is taken as an addend 3 times.
$3 \times 4 = 4 + 4 + 4 = 12$

4×3 means that 3 is taken as an addend 4 times.
$4 \times 3 = 3 + 3 + 3 + 3 = 12$

How can we extend this definition of multiplication to cover the set of integers? For example, what will we mean by $a \times b$ if "a" is a negative integer? Rather than talking about taking "b" as an addend "a" number of times, we will take "b" as a subtrahend "a" number of times. Then we have accounted for the fact that "a" is negative.

A. Product of Two Positive Integers

If "a" and "b" are positive integers, a $\times$ b means that "b" is taken as an addend "a" times.

EXAMPLE: $(+3) \times (+4) = ▨$

Solution

$(+3) \times (+4)$ means that $(+4)$ is taken as addend 3 times.

$$(+3) \times (+4) = (+4) + (+4) + (+4)$$
$$= +12$$

In the example above, note that the sum is positive. The absolute value of the product is 12, and it can be obtained by finding the product of the absolute values of the two numbers. $(3 \times 4 = 12)$

> **MULTIPLICATION - TWO POSITIVE INTEGERS**
>
> To multiply two positive integers, multiply their absolute values and give the product a positive sign.

Using the rule for multiplying two positive integers, find the following products.

(a) $(+8) \times (+4) = +32$ (c) $(+3) \times (+3) =$ ______

(b) $(+5) \times (+6) =$ ______ (d) $(+10) \times (+10) =$ ______

B. Product of Two Negative Integers

If "a" and "b" are negative integers, a $\times$ b means that "b" is taken as a subtrahend "a" times.

EXAMPLE: $(-3) \times (-4) = ▨$

Solution

$(-3) \times (-4)$ means that (-4) is taken as a subtrahend 3 times.

$$\left.\begin{aligned}(-3) \times (-4) &= -(-4) - (-4) - (-4) \\ &= +(+4) + (+4) + (+4)\end{aligned}\right\}$$

IN EACH TERM, CHANGE THE OPERATION TO ADDITION AND GIVE THE ADDITIVE INVERSE OF THE SUBTRAHEND.

$$= \boxed{+12}$$

In the example on page 30, note that the product is <u>positive</u>. The absolute value of the product is 12. It can be obtained by finding the product of the absolute values of the two numbers. $(3 \times 4 = 12)$

> **MULTIPLICATION - TWO NEGATIVE INTEGERS**
>
> To multiply two <u>negative</u> integers, multiply their absolute values and give the product a <u>positive</u> sign.

Using the rule for multiplying two negative integers, find the following products.

(a) $(-8) \times (-5) =$ +40 (c) $(-9) \times (-9) =$ ______

(b) $(-3) \times (-7) =$ ______ (d) $(-2) \times (-6) =$ ______

C. <u>Product of Two Integers with Unlike Signs</u>

If "a" is a positive integer and "b" is a negative integer, $a \times b$ means that "b" is taken as an <u>addend</u> "a" times.

EXAMPLE 1: $(+3) \times (-4) =$

<u>Solution</u>

$(+3) \times (-4)$ means that (-4) is taken as an <u>addend</u> 3 times.

$$(+3) \times (-4) = (-4) + (-4) + (-4)$$

$$= \boxed{-12}$$

If "a" is a negative integer and "b" is a positive integer, $a \times b$ means that "b" is taken as a <u>subtrahend</u> "a" times.

EXAMPLE 2: $(-3) \times (+4) =$

<u>Solution</u>

$(-3) \times (+4)$ means that $(+4)$ is taken as a <u>subtrahend</u> 3 times.

$$\left.\begin{aligned}(-3) \times (+4) &= -(+4) - (+4) - (+4) \\ &= +(-4) + (-4) + (-4)\end{aligned}\right\}$$

IN EACH TERM, CHANGE THE OPERATION TO ADDITION AND GIVE THE ADDITIVE INVERSE OF THE SUBTRAHEND.

$$= \boxed{-12}$$

In the two examples on page 31, note that the product is negative. The absolute value of the product is 12, and it can be obtained by finding the product of the absolute values of two numbers. ($3 \times 4 = 12$)

MULTIPLICATION - TWO INTEGERS WITH UNLIKE SIGNS
To multiply two integers with unlike signs, multiply their absolute values and give the product a negative sign.

Using the rule for multiplying two integers with unlike signs, find the following products.

$(-5) \times (+6) =$ -30 $(-2) \times (+11) =$ _____

$(+8) \times (-3) =$ _____ $(+9) \times (-9) =$ _____

Self-correcting Exercise #9

Answers may be found on page 51 of this lesson.

1. Fill in the blanks in the chart below. First tell if the signs of the two integers are alike or different. Then, tell whether the product will be positive or negative. In the last column, state the product.

		Are the signs alike or different?	Will the product be positive or negative?	Product
(a)	$(+2) \times (-5)$	different	negative	-10
(b)	$(-7) \times (-6)$			
(c)	$(+6) \times (+6)$			
(d)	$(-4) \times (+5)$			
(e)	$(-3) \times (-8)$			
(f)	$(+7) \times (-9)$			

D. Product of More Than Two Integers

The rules for multiplying integers can be extended to cover any number of factors.

PRODUCT OF TWO OR MORE INTEGERS
1. The absolute value of an indicated product of integers is the product of the absolute values of the integers.
2. An indicated product containing an odd number of negative factors is a negative integer.
3. An indicated product containing an even number of negative factors is a positive integer.

EXAMPLES:

PRODUCT IS NEGATIVE

$(-5)(-2)(+2)(-4)(+3) = -(|-5| \times |-2| \times |+2| \times |-4| \times |+3|)$

There are 3 negative factors. Since 3 is an odd number, the product must be negative.

$= -(5 \times 2 \times 2 \times 4 \times 3)$

$= -240$

PRODUCT IS POSITIVE

$(-2)(-1)(-1)(+3)(-4)(+2) = +(|-2| \times |-1| \times |-1| \times |+3| \times |-4| \times |+2|)$

There are 4 negative factors. Since 4 is an even number, the product must be positive.

$= +(2 \times 1 \times 1 \times 3 \times 4 \times 2)$

$= +48$

Self-correcting Exercise #10

Answers may be found on page 51 of this lesson.

1. Find the following products.

(a) $(-4)(-7)(+2)$ = ________ (b) $(-1)(+1)(-1)(+1)(-1)$ = ________

(c) $(-3)(-3)(-3)(-3)$ = ________ (d) $(-2)(-2)(-3)(-1)(-4)$ = ________

(e) $(-4)(-2)(-1)$ = ________ (f) $(+2)(+2)(-1)(+2)$ = ________

2. State whether each of the following products will be positive or negative.

	Positive or Negative?
(a) (-7)(-13)(+9)(+2)(-16)(+77)(-2)	__________
(b) (-2)(-2)(+8)(-2)(-8)(-3)	__________
(c) (-1)(-1)(-1)(-1)(-1)(-1)(-1)	__________
(d) (-1)(+2)(-3)(+4)(-5)(+6)(-7)(-8)(-9)	__________

E. Dividing Integers

Review the definition of division of whole numbers on page 27, Lesson 2. Division in set I can be defined in a similar manner.

> **DEFINITION OF DIVISION**
> **IN SET I**
>
> For any three integers a, b, and c,
> $a \div b = c$ if and only if $a = b \times c$.
>
> "Divide a by b" means "find the number which b must be multiplied by to make a."

EXAMPLE: $(-6) \div (-3) = \square$

Solution. We could change this division question to a multiplication question by asking ourselves, "What number must we multiply (-3) by to make (-6)?"

i.e. $(-3) \times \square = -6$

From our rules for multiplying integers, we know that

$$(-3) \times \boxed{+2} = -6$$

Changing back to a division question, we can say that

$$(-6) \div (-3) = \boxed{+2}.$$

Thus, the quotient of two negative integers is a positive integer.

Using this same approach, find the following quotients.

(+8) ÷ (+2) = ____	(-12) ÷ (-4) = _____
(-16) ÷ (+4) = _____	(-32) ÷ (+8) = _____
(+24) ÷ (-6) = _____	(-4) ÷ (-1) = _____

DIVIDING TWO INTEGERS

1. If the signs of the two integers are alike, divide their absolute values and give the quotient a positive sign.

   e.g. (+8) ÷ (+4) = +2
   (-8) ÷ (-4) = +2

2. If the signs of the two integers are different, divide their absolute values and give the quotient a negative sign.

   e.g. (-8) ÷ (+4) = -2
   (+8) ÷ (-4) = -2

<u>Self-correcting Exercise #11</u>

Answers may be found on page 52 of this lesson.

1. Fill in the blanks in the following table.

		Are the signs alike or different?	Is the quotient pos. or neg.?	Quotient
(a)	(+16) ÷ (+2)	alike	positive	+8
(b)	(-15) ÷ (+3)			
(c)	(-12) ÷ (-4)			
(d)	(+6) ÷ (+2)			
(e)	(+24) ÷ (-8)			
(f)	(-49) ÷ (-7)			

EXERCISE - Multiplying and Dividing Integers

1. Fill in the blanks.

(a) When you multiply two integers:

(i) the product is positive if both integers are ________________ or both integers are ________________.

(ii) the product of a positive integer and a negative integer is a ________________ integer.

(iii) the absolute value of the product is the ________________ of the ________________ values of the integers.

(b) A product containing an even number of negative factors will be a ________________ integer.

(c) The quotient of two positive or two negative integers is a ________________ integer.

2. Simplify each of the following expressions.

(a) $(+144) \div (-8) =$ __________ (b) $(-13) \times (-4) =$ __________

(c) $(-1) \times (+7) =$ __________ (d) $(-33) \div (+3) =$ __________

(e) $(-17) \times (-2) =$ __________ (f) $(-192) \div (-8) =$ __________

(g) $(+66) \div (+22) =$ __________ (h) $(+5) \times (-15) =$ __________

(i) $(-1) \times (-1) \times (-1) \times (-1) =$ __________

(j) $(-2) \times (+3) \times (-4) \times (+2) \times (-1) =$ __________

(k) $[(-5) \times (-8)] \div (+2) =$ (______) $\div (+2) =$ __________

(l) $(-4)\,[(-18) \div (+3)] = (-4) \times$ (______) $=$ __________

(m) $[(-32) \div (-4)] \div (-2) =$ (______) $\div (-2) =$ __________

Topic Five: Writing Positive Integers Without Their Signs

Expressions involving integers are far less cumbersome to work with if all positive integers are written without their signs. That is, an integer like +7 can be written as the natural number 7. Whenever a number is written with no sign in front, you must assume it is a positive number.

Note how unnecessary positive signs can be dropped in the following expressions:

1. (+2) + (+5) represents the sum of positive 2 and positive 5. It could be written in simpler form as:

   $$2 + 5$$

2. (-7) + (+2) represents the sum of negative 7 and positive 2. It could be written in simpler form as:

   $$(-7) + 2$$

3. (-9) × (+3) represents the product of negative nine and positive 3. It could be written in simpler form as:

   $$(-9) \times 3$$

How would you write (+12) ÷ (+3) in simpler form? ____________________

Even though the positive signs of quality are not usually written, you must remember that they are still implied. Note how the following expressions are interpreted.

1. (-8) + 6 represents the sum of <u>negative</u> 8 and <u>positive</u> 6. In order to find this sum, we must use the rule for adding two integers with unlike signs. (We subtract the two absolute values and attach the sign of the number with the larger absolute value.)

   $$(-8) + 6 = -2$$

2. 7 × (-5) represents the product of <u>positive</u> 7 and <u>negative</u> 5. Since the signs are unlike, the product will be negative.

   $$7 \times -5 = -35$$

3. (-8) ÷ 2 represents the quotient of __________ 8 and __________ 2.

   Since the signs are __________, the quotient will be __________________.

   $$(-8) \div 2 = \underline{\qquad}$$

4. 3 + (-6) represents the sum of __________ 3 and __________ 6. In order to find the sum, we must __________ the two absolute values and attach the sign of the number with the __________ absolute value.

   $$3 + (-6) = \underline{\qquad}$$

When you are evaluating an expression involving integers and your answer is a positive number, you need not write down the plus sign in front of the number.

EXAMPLE: $8 + (-3) = \boxed{}$

Solution

In order to find this sum, you must use the rule for adding integers with unlike signs. The absolute value of the sum is 5 since $8 - 3 = 5$. The sum is positive since the number with the larger absolute value is positive. The answer is +5, but it is easier to write this as 5.

$$8 + (-3) = \boxed{5}$$

Find the following sums. If the answer is positive, omit the sign.

1. $3 + 7 =$ _____	3. $7 + (-2) =$ _____
2. $-8 + 9 =$ _____	4. $(-12) + 7 =$ _____

Don't be confused when the positive signs of quality are omitted in subtraction questions. You still must change the operation to addition and give the additive inverse of the second number. Just remember that the positive signs of quality are implied even though they are not written. Note how the following subtraction questions are interpreted.

1. 3 - 7 represents the difference of positive 3 and positive 7. In order to find this difference, we must change the operation to addition and give the additive inverse of positive 7.

   CHANGE OPERATION TO ADDITION.
   ADDITIVE INVERSE OF 7.

   i.e. $3 - 7 = 3 + (-7)$

   Now, apply the rule for adding two integers with unlike signs.

   $= -4$

2. 6 - (-2) represents the difference of positive 6 and negative 2. Change the operation to addition and give the additive inverse of negative 2.

   CHANGE OPERATION TO ADDITION.

   $6 - (-2) = 6 + (+2)$ ADDITIVE INVERSE OF -2.

   Now, you must find the sum of two positive integers.

   $= 8$

3. (-8) - 3 represents the difference of __________ 8 and __________ 3.

Change the operation to addition and give the additive inverse of positive 3.

$$(-8) - 3 = (-8) + (-3)$$

Now, you must find the sum of two __________ integers.

= _____

Self-correcting Exercise #12

Answers may be found on page 52 of this lesson.

1. In each expression, first tell whether the signs of the two integers are alike or different. Then decide if the result will be positive or negative. In the last column, give the result. (Omit sign if result is positive.)

		Are the signs alike or different?	Will the result be pos. or neg.?	Result
(a)	16 ÷ (-4)	different	negative	-4
(b)	3 × 5	____	____	____
(c)	8 + (-5)	____	____	____
(d)	9 + 3	____	____	____
(e)	2 × (-7)	____	____	____
(f)	(-6) + 3	____	____	____
(g)	18 ÷ 6	____	____	____
(h)	(-1) × 3	____	____	____
(i)	(-4) ÷ 2	____	____	____

2. Rewrite each subtraction question as an addition question. Then decide if the signs of the two integers in the sum are alike or different. In the last column, give the sum.

		Corresponding addition question	Are the signs alike or different?	Sum
(a)	6 - 12	6 + (-12)	different	-6
(b)	-4 - 2	______	______	____
(c)	7 - (-9)	______	______	____
(d)	13 - 17	______	______	____
(e)	-3 - (-4)	______	______	____
(f)	-9 - (-1)	______	______	____

3. Simplify each expression. (Simplify inside brackets first.)

(a) (-6 + 9) - 7

= 3 - ____

= ____ + (____) } CORRESPONDING ADDITION QUESTION

= ____

(b) (2 × -3) - 4

= ____ - ____

= ____ + ____ } CORRESPONDING ADDITION QUESTION

= ____

(c) -2(3 - 7)

= ____ [____ + (____)]

= ____ × ____

= ____

(d) -8 + (-2 - 4)

= -8 + [____ + (____)]

= -8 + (____)

= ____

EXERCISE - Operating With Integers

1. Simplify each expression.

(a) $6 - 3 = $ 3

(b) $4 - 8 = 4 + (-8) = $ ____

(c) $-3 + 4 = $ ____

(d) $-2 + 2 = $ ____

(e) $-5 + 7 = $ ____

(f) $-14 - 14 = $ ____ $+ (-14) = $ ____

(g) $-33 + 10 = $ ____

(h) $14 + 14 = $ ____

(i) $4 - 14 = $ ____ $+ (-14) = $ ____

(j) $-4 - 14 = $ ________ = ____

(k) $-4 + 14 = $ ____

(l) $3 - 7 = $ ________ = ____

(m) $-8 + 12 = $ ____

(n) $-16 + 3 = $ ____

(o) $8 - (-2) = 8 + ($ ____ $) = $ ____

(p) $(-12) - (-7) = $ ____ $+ (+7) = $ ____

(q) $6 - 9 = $ ____ $+ ($ ____ $) = $ ____

(r) $(-6) - (-8) = $ ________ = ____

(s) $8 \times -9 = $ ____

(t) $-4 \times -4 = $ ____

(u) $-1 \times 6 = $ ____

(v) $6 \div -3 = $ ____

(w) $-20 \div 4 = $ ____

(x) $(-14) \div (-7) = $ ____

(y) $5 \times -2 \times 3 \times -4 = $ ____

(z) $-2 \times -2 \times -2 \times -2 = $ ____

2. Simplify each expression until your final answer is a single integer. (Simplify inside brackets first.)

(a) $(3 \times 6) - (4 \times -7)$

$=$ ____ $-$ (____)

$=$ ____ $+$ (____) CORRESPONDING ADDITION QUESTION

$=$ ____

(b) $4[(-6) - (-3)]$ CORRESPONDING ADDITION QUESTION

$= 4[($ ____ $) + ($ ____ $)]$

$= 4 \times ($ ____ $)$

$=$ ____

(c) $(3 \times -5) - 4$

$= ($ ____ $) -$ ____

$= ($ ____ $) + ($ ____ $)$

$=$ ____

(d) $(-8 \times -4) \div -2$

$=$ ____ $\div -2$

$=$ ____

(e) $(3 \times 2) - (4 \times 5)$

$= \_\_\_\_ - \_\_\_\_$

$\left.\begin{array}{l}= \_\_\_\_ + (\_\_\_\_)\end{array}\right\}$ CORRESPONDING ADDITION QUESTION

$= \_\_\_\_$

(f) $3 + (-5 - 1)$

$= \_\_\_\_ + [(\_\_\_) + (\_\_\_)]$

$= \_\_\_\_ + (\_\_\_\_)$

$= \_\_\_\_$

(g) $4 (3 - 8)$

$= 4 \times [\_\_\_\_ + (\_\_\_\_)]$

$= 4 \times (\_\_\_\_)$

$= \_\_\_\_$

(h) $(-2 + 5) - 8$

$= \_\_\_\_ - \_\_\_\_$

$= \_\_\_\_ + (\_\_\_)$

$= \_\_\_\_$

Topic Six: Properties of the Integers

In Lesson 2, you learned various properties that applied to the set of whole numbers under the two fundamental operations of addition and multiplication. Upon inspection, we find that the set of integers satisfies all the properties that we have already attributed to the whole numbers, besides having some new properties of its own.

Fill in any blanks that appear below.

A. Commutative Properties

Addition and multiplication are commutative in Set I, but subtraction and division are not.

EXAMPLES:

1. $(-2) + 5 = 5 + (-2)$ since both sides equal 3.
2. $3 \times (-6) = (-6) \times 3$ since both sides equal _____.
3. $2 - 5 \neq 5 - 2$ ← SYMBOL FOR "DOES NOT EQUAL". since the left side equals _____ and the right side equals _____.
4. $-8 \div 2 \neq 2 \div -8$ because the left side equals _____ and the right side is not an integer.

B. Associative Properties

Addition and multiplication are associative in set I, but subtraction and division are not.

EXAMPLES:

1. $(-12 + 3) + 7 = -12 + (3 + 7)$ since both sides equal _____.

2. $(3 \times -2) \times 4 = 3 \times (-2 \times 4)$ since both sides equal _____.

3. $(2 - 6) - 4 \neq 2 - (6 - 4)$ since the left side equals _____ and the right side equals _____.

4. $(-8 \div 4) \div - 2 \neq -8 \div (4 \div -2)$ since the left side equals _____ and the right side equals _____.

C. Distributive Properties

Multiplication distributes over both addition and subtraction in set I. (You will recall that in set W, multiplication only distributes over subtraction if the subtraction yields a whole number.)

EXAMPLE 1: In order to illustrate that multiplication distributes over addition, show that $-3(-2 + 6) = (-3 \times -2) + (-3 \times 6)$.

Solution

Left Side $= -3(-2 + 6)$	Right Side $= (-3 \times -2) + (-3 \times 6)$
$= -3 \times$ _____	$=$ _____ $+$ _____
$=$ _____	$=$ _____

Are the two sides equal? _____

EXAMPLE 2: In order to illustrate that multiplication distributes over subtraction, show that $-3(5 - 9) = (-3 \times 5) - (-3 \times 9)$

Solution

Left Side $= -3(5 - 9)$	Right Side $= (-3 \times 5) - (-3 \times 9)$
$= -3 \times$ _____	CORRESPONDING $=$ _____ $-$ _____
$=$ _____	ADDITION $\rightarrow =$ _____ $+$ _____
	QUESTION $=$ _____

Are the two sides equal? _____

D. Closure Properties

Like set W, set I is closed under addition and multiplication but not under division. But, set I has the added feature that it is also closed under subtraction. Thus, closure under subtraction is a new property of the set of integers.

Since set I is closed under addition, multiplication, and subtraction, this means that when we add, multiply, or subtract two integers, the result will always be an integer. Simplify the following expressions and note that the results are always integers.

$-5 + 2 =$ ______	$6 - 3 =$ ______	$9 \times -7 =$ ______
$-4 \times -2 =$ ______	$7 - 9 =$ ______	$(-3) + (-4) =$ ______

Set I is not closed under division since the quotient of two integers is not always an integer. Write N after each of the following quotients in which the result is not an integer.

$4 \div 7$ N	$-16 \div -4$ ______
$8 \div -2$ ______	$-14 \div -16$ ______
$-2 \div 8$ ______	$14 \div 5$ ______

E. Identity Elements

Like set W, set I has identity elements for both addition and multiplication. The additive identity is zero since when zero is added to any integer, that integer is left unchanged.

EXAMPLES:

$-8 + 0 =$ ______
______ $+ (-3) = -3$
$0 +$ ______ $= 6$

Fill in the blanks.

The multiplicative identity is one since when any integer is multiplied by one, that integer is left unchanged.

EXAMPLES:

$1 \times -4 =$ ______
___ $\times 3 = 3$
___ $\times 1 = -6$

Fill in the blanks.

F. Inverse Elements

In Lesson 2, when we talked about the properties of set W, no mention was made of inverse elements. In set I, we have a new property concerning additive inverses.

Every integer has an inverse element under the operation of addition. The sum of a number and its additive inverse is zero.

EXAMPLES:

The additive inverse of -6 is 6 since -6 + 6 = 0.

The additive inverse of 4 is _____ since 4 + ___ = 0.

The additive inverse of -8 is _____ since ___ + ___ = ___.

G. Summary - Properties of the Integers Under Addition and Multiplication

For any three integers a, b, and c:

	Addition	Multiplication
Closure properties	$a + b$ is an integer.	ab is an integer.
Commutative properties	$a + b = b + a$	$ab = ba$
Associative properties	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Distributive property	$a(b + c) = ab + ac$	
Identity elements	Additive identity is zero. $a + 0 = a$	Multiplicative identity is one. $a \times 1 = a$
Inverse elements	*Additive inverse of a is -a. $a + (-a) = 0$	

*This is a property of the integers but not of the whole numbers.

EXERCISE - Properties of Set I

1. Fill in the blanks.

   (a) Like set W, set I is closed under the operations of ________________ and ________________, but is not closed under the operation of ________________.

   (b) Set I is closed under the operation of ________________ while set W is not.

   (c) Since $a + 0 = a$, zero is called the ________________ ________________ for set I.

   (d) Since $a \times 1 = a$, one is called the ________________ ________________ for set I.

   (e) The operations of ________________ and ________________ are associative and commutative in set I, but the operations of ________________ and ________________ are not.

   (f) In set I, multiplication distributes over both ________________ and ________________.

2. Name the property that justifies each statement. Choose from the following list.

LIST	
associative prop. of addition	closure prop. of addition
associative prop. of mult.	closure prop. of mult.
commutative prop. of addition	additive identity property
commutative prop. of mult.	mult. identity property
distributive property	additive inverse property

	Property
(a) $1 \times -7 = -7$	mult. identity property
(b) $-8 \times 5 = 5 \times -8$	________________
(c) $(-5) + 4 = 4 + (-5)$	________________

(d) $(-6 + 7)$ is an integer. ______________________

(e) $(-8 + 7) + 9 = -8 + (7 + 9)$ ______________________

(f) $(6 \times -4) + (3 \times -4) = (6 + 3)(-4)$ ______________________

(g) (-3×-5) is an integer. ______________________

(h) $(-5) + 5 = 0$ ______________________

(i) $1 \times 3 \times -5 = 3 \times -5$ ______________________

(j) $(-2)(m - 3) = (m - 3)(-2)$ ______________________

(k) $-8 + 3 + 0 = -8 + 3$ ______________________

(l) $(6 \times -5) \times 4 = 6 \times (-5 \times 4)$ ______________________

Key to Self-correcting Exercises in Lesson 3

Exercise #1, page 5

1. (a) $|-7| \times |-3| = 7 \times 3 = \underline{21}$

   (b) $|8 - 5| = |3| = \underline{3}$

   (c) $|-10| + |5| = 10 + 5 = \underline{15}$

   (d) $|-16| \div |4| = 16 \div 4 = \underline{4}$

2. (a) $|-14|$ is greater since 14 is greater than 2.
   (b) $|-7|$ is greater since 7 is greater than 2.
   (c) $|13|$ is greater since 13 is greater than 12.
   (d) $|-6|$ is greater since 6 is greater than 5.
   (e) $|-2|$ is greater since 2 is greater than 0.
   (f) $|-5|$ is greater since 5 is greater than 4.

Exercise #2, page 7

1. (a) false; -2 lies to the right of -6, so $-2 > -6$.
   (b) true; -5 lies to the left of -4.
   (c) true; -4 lies to the left of 2.
   (d) false; 0 lies to the left of 3, so $0 < 3$.
   (e) true; 6 lies to the right of -2.
   (f) true; 12 lies to the right of -14.
   (g) true; -8 lies to the left of zero.
   (h) false; -12 lies to the left of -6, so $-12 < -6$.

2. (a) $-1 > -8$ (−1 lies to the <u>right</u> of −8.)
   (b) $7 > 3$ (7 lies to the <u>right</u> of 3.)
   (c) $-7 < -3$ (−7 lies to the <u>left</u> of −3.)
   (d) $-5 < 7$ (−5 lies to the <u>left</u> of 7.)
   (e) $0 > -3$ (0 lies to the <u>right</u> of −3.)
   (f) $8 > -12$ (8 lies to the <u>right</u> of −12.)

Exercise #3, page 8

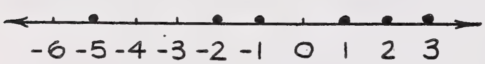
1. (a) $-7 \leqq x \leqq -2$ (b) $x < -5$ (c) $x \geqq -10$
   (d) $-3 < x < 7$ (e) $-100 < x \leqq -50$ (f) $-4 < x < 3$

SMALLER NUMBER APPEARS ON THE LEFT.

2. (a) $\{x \mid x \geqq -5,\ x \in I\}$; $\{-5, -4, -3, \ldots\}$
   (b) $\{x \mid -100 \leqq x \leqq 50,\ x \in I\}$; $\{-100, -99, -98, \ldots, 50\}$
   (c) $\{x \mid -13 < x \leqq -6,\ x \in I\}$; $\{-12, -11, -10, -9, -8, -7, -6\}$

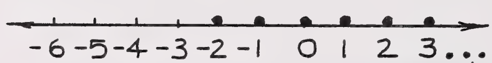
Exercise #4, page 10

1. (a) $A = \{3, 1, -2, -1, -5, 2\}$



The graph consists of 6 points.

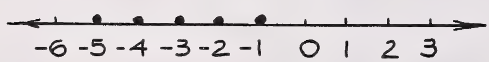
   (b) $B = \{-2, -1, 0, 1, \ldots\}$



The graph consists of all the points to the right of -3 that represent integers.

DON'T FORGET THESE DOTS.

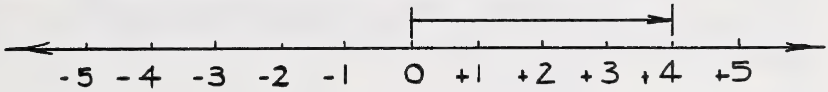
   (c) $C = \{-5, -4, -3, -2, -1\}$



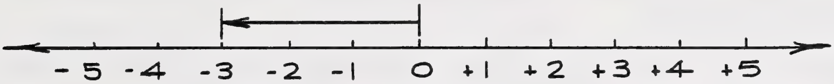
The graph consists of 5 points.

Exercise #5, page 16

1. (a) The integer 4 can be represented by a trip of 4 units to the right. The ray begins at 0 and ends at 4.



(b) The integer -3 can be represented by a trip of 3 units to the left. The ray begins at 0 and ends at -3.

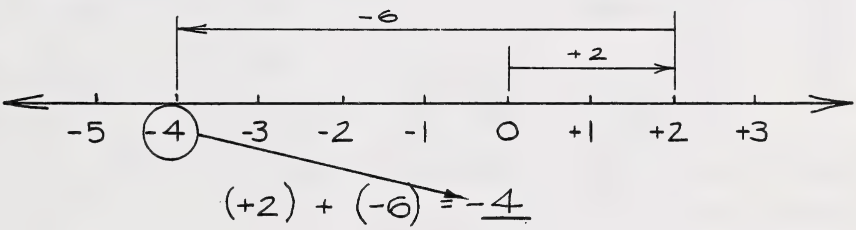


2. (a) The integer -2 corresponds to this trip. (The integer is negative since the trip is to the left. The magnitude of the integer is 2 since the trip covers 2 spaces.)

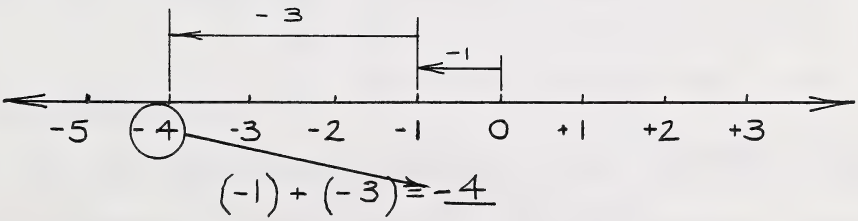
(b) The integer +3 corresponds to this trip. (The integer is positive since the trip is to the right. The magnitude of the integer is 3 since the trip covers 3 spaces.)

Exercise #6, page 17

1. (a)



(b)



2. (a) The first trip is 2 units to the left, so it represents the integer -2. The second trip is 4 units to the right, so it represents the integer +4. The second trip ends at +2, so the sum is +2.

$$\underline{(-2)} + \underline{(+4)} = \underline{+2}$$

Exercise #6, cont'd

2\. (b) The first trip is 5 units to the left, so it represents the integer -5. The second trip is 4 units to the right, so it represents the integer +4. The second trip ends at -1, so the sum is -1.

$$\underline{(-5)} + \underline{(+4)} = \underline{-1}$$

Exercise #7, page 22

1\. In each case, add the absolute values and attach the common sign to the answer.

(a) $(+15) + (+6) = \underline{+21}$ (b) $(+1) + (+99) = \underline{+100}$
(c) $(-3) + (-16) = \underline{-19}$ (d) $(+30) + (+20) = \underline{+50}$
(e) $(-12) + (-12) = \underline{-24}$ (f) $(-6) + (-4) = \underline{-10}$

2\. In each case, subtract the smaller absolute value from the larger and attach the sign of the integer with the larger absolute value.

(a) $(+12) + (-9) = +3$ (b) $(+17) + (-25) = -8$
(c) $(-35) + (+15) = -20$ (d) $(-13) + (+19) = +6$
(e) $(-16) + (+20) = +4$ (f) $(-11) + (+11) = 0$

3\. (a) -2 (b) -3 (c) +6 (d) -11 (e) +2
(f) +9 (g) +13 (h) -37 (i) 0 (j) +17

4\. (a) $[(+8) + (-2)] + (-9) = (+6) + (-9) = \underline{-3}$
(b) $[(-3) + (-2)] + (+1) = (-5) + (+1) = \underline{-4}$
(c) $[(+4) + (+7)] + (-3) = (+11) + (-3) = \underline{+8}$
(d) $[(-7) + (+4)] + (-8) = (-3) + (-8) = \underline{-11}$
(e) $[(-5) + (-2)] + (-6) = (-7) + (-6) = \underline{-13}$

Exercise #8, page 27

CHANGE OPERATION TO ADDITION.
CHANGE +4 TO -4.

1\. (a) $(+8) - (+4) = \underline{(+8) + (-4)}$ (d) $(+1) - (-4) = \underline{(+1) + (+4)}$

(b) $(-3) - (-3) = \underline{(-3) + (+3)}$ (e) $(+2) - (+9) = \underline{(+2) + (-9)}$

(c) $(+4) - (+10) = \underline{(+4) + (-10)}$ (f) $(-6) - (+3) = \underline{(-6) + (-3)}$

2.

		Corresponding addition question	Are the signs alike or different?	Sum
(a)	(+13) - (-9)	(+13) + (+9)	alike	+22
(b)	(-6) - (+7)	(-6) + (-7)	alike	-13
(c)	(-3) - (-5)	(-3) + (+5)	different	+2
(d)	(-4) - (+5)	(-4) + (-5)	alike	-9
(e)	(+23) - (+7)	(+23) + (-7)	different	+16

<u>Exercise #9, page 32</u>

1.

		Are the signs alike or different?	Will the product be positive or negative?	Product
(a)	(+2) × (-5)	different	negative	-10
(b)	(-7) × (-6)	alike	positive	+42
(c)	(+6) × (+6)	alike	positive	+36
(d)	(-4) × (+5)	different	negative	-20
(e)	(-3) × (-8)	alike	positive	+24
(f)	(+7) × (-9)	different	negative	-63

<u>Exercise #10, page 33</u>

1. (a) +56 (even no. of neg. factors) (b) -1 (odd no. of neg. factors)

   (c) +81 (even no. of neg. factors) (d) -48 (odd no. of neg. factors)

   (e) -8 (odd no. of neg. factors) (f) -8 (odd no. of neg. factors)

2. (a) positive; since there are 4 negative factors. (4 is even.)
   (b) negative; since there are 5 negative factors. (5 is odd.)
   (c) negative; since there are 7 negative factors. (7 is odd.)
   (d) positive; since there are 6 negative factors. (6 is even.)

Exercise #11, page 35

1.

		Are the signs alike or different?	Is the quotient pos. or neg.?	Quotient
(a)	(+16) ÷ (+2)	alike	positive	+8
(b)	(-15) ÷ (+3)	different	negative	-5
(c)	(-12) ÷ (-4)	alike	positive	+3
(d)	(+6) ÷ (+2)	alike	positive	+3
(e)	(+24) ÷ (-8)	different	negative	-3
(f)	(-49) ÷ (-7)	alike	positive	+7

Exercise #12, page 39

1.

		Are the signs alike or different?	Will the result be pos. or neg.?	Result
(a)	16 ÷ (-4)	different	negative	-4
(b)	3 × 5	alike	positive	15
(c)	8 + (-5)	different	positive	3
(d)	9 + 3	alike	positive	12
(e)	2 × (-7)	different	negative	-14
(f)	(-6) + 3	different	negative	-3
(g)	18 ÷ 6	alike	positive	3
(h)	(-1) × 3	different	negative	-3
(i)	(-4) ÷ 2	different	negative	-2

2.

		Corresponding addition question	Are the signs alike or different?	Sum
(a)	6 - 12	6 + (-12)	different	-6
(b)	-4 - 2	-4 + (-2)	alike	-6
(c)	7 - (-9)	7 + (+9)	alike	16
(d)	13 - 17	13 + (-17)	different	-4
(e)	-3 - (-4)	-3 + (+4)	different	1
(f)	-9 - (-1)	-9 + (+1)	different	-8

3. (a) $(-6 + 9) - 7$

$= 3 - 7$

$= 3 + (-7)$

$= -4$

(b) $(2 \times -3) - 4$

$= -6 - 4$

$= -6 + (-4)$

$= -10$

(c) $-2(3 - 7)$

$= -2\ [3 + (-7)]$

$= -2 \times -4$

$= 8$

(d) $-8 + (-2 - 4)$

$= -8 + [-2 + (-4)]$

$= -8 + (-6)$

$= -14$

End of Lesson 3

Lesson 4

Rational Numbers

Basic Algebra and Geometry

RATIONAL NUMBERS

Topic One: The Set of Rational Numbers

In Lesson 3, you learned that the set of integers

$$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

is not closed under division. This means that the quotient of two integers is not always an integer. In fact, the quotient

$$a \div b$$

represents an integer only if $|b|$ divides evenly into $|a|$, with no remainder.

EXAMPLES:

1. The quotient $(8 \div -2)$ belongs to set I since 2 divides evenly into 8, with no remainder.

2. The quotient $(-7 \div -3)$ does not belong to set I since 3 does not divide evenly into 7.

3. The quotient $(4 \div 5)$ does not belong to set I since 5 does not divide evenly into 4.

Beside each quotient below, write I if it represents an integer.

1. $-16 \div -4$ I
2. $8 \div -3$ ____
3. $-3 \div 7$ ____
4. $0 \div 4$ ____
5. $20 \div -5$ ____
6. $-5 \div -5$ ____
7. $6 \div 4$ ____
8. $-2 \div 6$ ____

The SET OF RATIONAL NUMBERS was invented in order to obtain closure under division (except for division by zero). This new set also helps us refer to parts of a whole and points on the number line between two consecutive integers.

A. Definition of Rational Numbers

The set of rational numbers contains elements which can be represented as quotient of two integers, such as:

$$\frac{3}{4}, \quad -4\frac{1}{4} \left(\text{or } \frac{-17}{4}\right), \quad 5 \left(\text{or } \frac{5}{1}, \text{ or } \frac{10}{2}, \text{ or } \frac{15}{3}, \text{ etc.}\right), \quad \frac{-7}{-5}$$

We previously would have written these quotients as:

$3 \div 4$, $-17 \div 4$, $5 \div 1$ (or $10 \div 2$ or $15 \div 3$, etc.), $-7 \div -5$

In our new notation, a horizontal bar replaces the familiar division sign, "÷".

In general, for any two integers a and b, where $b \neq 0$, there is a numeral $\frac{a}{b}$ (read "a over b") which represents the quotient $(a \div b)$.

The symbol $\frac{a}{b}$ is called a FRACTION; a is the NUMERATOR, b is the DENOMINATOR, and a and b are called the TERMS of the fraction. The denominator of a fraction can never be zero since division by zero is impossible. Thus, fractions like $\frac{-3}{0}$, $\frac{8}{0}$, $\frac{15}{0}$ do not exist since their denominators are zero. Give examples of three other fractions that do not exist because of this reason.

_____, _____, _____

State the numerator and denominator of each of the fractions below.

1. $\frac{6}{-11}$ Numerator is _____. Denominator is _____.

2. $\frac{-7}{-9}$ Numerator is _____. Denominator is _____.

3. $\frac{0}{8}$ Numerator is _____. Denominator is _____.

The set of integers is a SUBSET of the set of rational numbers. This means that every integer is a rational number and hence can be written in the rational form $\frac{a}{b}$.

Any rational number in which the denominator divides evenly into the numerator, with no remainder, represents an integer.

EXAMPLES:

1. $\frac{9}{-3}$ represents the quotient $(9 \div -3)$ and equals the integer -3.

2. $\frac{0}{-5}$ represents the quotient $(0 \div -5)$ and equals the integer 0.

   In general, any rational number whose numerator is 0 equals the integer 0 (provided the denominator is not 0).

Beside each rational number below, write I if it represents an integer.

1. $\frac{4}{3}$ ____ 2. $\frac{12}{-4}$ I 3. $\frac{0}{-11}$ ____

4. $\frac{-15}{-3}$ ____ 5. $\frac{-2}{4}$ ____ 6. $\frac{18}{2}$ ____

Give an integer that is equivalent to each rational number below.

1. $\frac{5}{-5} = -1$ 2. $\frac{-33}{-3} =$ ____ 3. $\frac{14}{2} =$ ____

(SINCE $5 \div -5 = -1$)

4. $\frac{0}{2} =$ ____ 5. $\frac{-24}{6} =$ ____ 6. $\frac{-7}{-7} =$ ____

Give four rational numbers that are equivalent to each integer.

1. 1 $\frac{8}{8}$, $\frac{-2}{-2}$, ____, ____ 2. -5 $\frac{-35}{7}$, ____, ____, ____

(SINCE $8 \div 8 = 1$)

3. -1 $\frac{-12}{12}$, ____, ____, ____ 4. 12 $\frac{-12}{-1}$, ____, ____, ____

5. 0 $\frac{0}{3}$, ____, ____, ____ 6. 4 $\frac{36}{9}$, ____, ____, ____

The infinite set of rational numbers cannot be tabulated in an orderly fashion so it is usually written in set-builder notation as follows:

$$Q = \{\frac{a}{b} | a, b \in I, \ b \neq 0\}.$$

This is read, "Q is the set of all numerals of the form a over b, such that a and b are integers and b does not equal zero." Note that, for convenience, we will always designate this set by the capital letter Q.

Is -6 a rational number? yes Is $\frac{1}{2}$? _____ Is 10? _____

Is $\frac{-2}{-3}$? _____ Is $\frac{5}{2}$? _____ Is $\frac{0}{3}$? _____ Is $\frac{3}{0}$? _____

B. Positive and Negative Rational Numbers

The positive integer 2 can be represented by any rational number in which the denominator divides into the numerator twice, with no remainder. Thus, the positive integer 2 can be written in the following equivalent rational forms:

$$\frac{2}{1}, \quad \frac{-2}{-1}, \quad \frac{4}{2}, \quad \frac{-4}{-2}, \quad \frac{6}{3}, \quad \frac{-6}{-3}, \text{ and so on.}$$

Note that each of these rational forms of 2 has two positive terms or two negative terms. (i.e. the signs of the terms are alike.)

In general, any positive rational number can be written in the form $\frac{a}{b}$ or $\frac{-a}{-b}$, where a and b are natural numbers. To cut down on the number of signs of quality that appear in a rational number, numerals of the form $\frac{-a}{-b}$ are generally rewritten in the form $\frac{a}{b}$.

EXAMPLES:

$\frac{-5}{-6}$ is usually written as $\frac{5}{6}$.

$\frac{-9}{-7}$ is usually written as $\frac{9}{7}$.

$\frac{5}{6}$ is left unchanged.

How would you write $\frac{-1}{-5}$? _____ $\frac{-3}{-5}$? _____ $\frac{-6}{-5}$? _____

$\frac{2}{3}$? _____ $\frac{-2}{-3}$? _____

The <u>negative</u> integer -2 can be represented by a rational number in which the denominator divides into the numerator evenly, giving a quotient of -2. Thus, the negative integer -2 can be written in the following equivalent rational forms:

$$\frac{-2}{1}, \quad \frac{2}{-1}, \quad \frac{-4}{2}, \quad \frac{4}{-2}, \quad \frac{-6}{3}, \quad \frac{6}{-3}, \text{ and so on.}$$

Note that each of these rational forms of -2 has one positive term and one negative term. (i.e. the signs of the terms are unlike.)

In general, any <u>negative</u> rational number can be written in the form $\frac{-a}{b}$ or $\frac{a}{-b}$ where a and b are natural numbers.

To simplify notation, negative rational numbers are generally written in the form $\frac{-a}{b}$. In other words, the minus sign is generally placed in the <u>numerator</u> of the fraction.

EXAMPLES:

$\frac{5}{-6}$ is usually written as $\frac{-5}{6}$.

$\frac{9}{-7}$ is usually written as $\frac{-9}{7}$.

$\frac{-5}{6}$ is left unchanged.

How would you write $\frac{5}{-8}$? _____ $\frac{-5}{8}$? _____ $\frac{10}{-11}$? _____

$\frac{4}{-7}$? _____ $\frac{-1}{3}$? _____ $\frac{1}{-3}$? _____

Self-correcting Exercise #1

Answers to this exercise may be found on page 52 of this lesson.

1. Write each quotient in rational form. Then write the rational number in the form $\frac{a}{b}$ or $\frac{-a}{b}$ where a and b are natural numbers.

	Quotient	Rational Form	Simpler Rational Form
(a)	$(-5) \div (-3)$	$\frac{-5}{-3}$	$\frac{5}{3}$
(b)	$2 \div (-3)$	________	________
(c)	$(-8) \div (-9)$	________	________
(d)	$3 \div (-5)$	________	________

2. Decide if each of the following numbers are positive, negative, zero, or undefined.

(a) $\frac{-3}{4}$ ____________________

(b) $\frac{0}{5}$ ____________________

(c) $\frac{-3}{-8}$ ____________________

(d) $\frac{4}{5}$ ____________________

(e) $\frac{6}{0}$ ____________________

(f) $\frac{2}{-3}$ ____________________

C. Equal Rational Numbers

You will recall that the integer -2 can be written in a variety of equivalent rational forms.

i.e. $-2 = \frac{-2}{1} = \frac{2}{-1} = \frac{-4}{2} = \frac{4}{-2} = \frac{-6}{3} = \frac{6}{-3}$ and so on.

If we compare any two of these rational forms of -2, we will find that the cross products of their terms are equal.

EXAMPLES:

1. $\frac{-2}{1}$ and $\frac{2}{-1}$ are equal rational numbers.

$$\frac{-2}{1} \times \frac{2}{-1}$$

The cross products are equal.
$(-2)(-1) = (1)(2)$

2. $\frac{-2}{1}$ and $\frac{-6}{3}$ are equal rational numbers.

$$\frac{-2}{1} \times \frac{-6}{3}$$

The cross products are equal.
$(-2)(3) = (1)(-6)$

EQUAL RATIONAL NUMBERS

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$,

$\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

For each pair of fractions below, find the cross products of the terms. Then, decide whether or not the fractions are equal.

Fraction Pair	Cross Products	Equal or Not Equal?
$\frac{4}{3}$, $\frac{12}{9}$	4x9=36 3x12=36	equal
$\frac{6}{8}$, $\frac{4}{6}$		
$\frac{-20}{16}$, $\frac{-5}{4}$		
$\frac{0}{5}$, $\frac{0}{7}$		
$\frac{6}{8}$, $\frac{-3}{2}$		

For a given rational number, how can we obtain another rational number that is equal to it? From our previous discussion, we know that $\frac{-2}{1}$ and $\frac{6}{-3}$ are equal rational numbers. We can change $\frac{-2}{1}$ to the equal rational number $\frac{6}{-3}$ if we <u>multiply</u> both its numerator and denominator by -3.

$$\text{i.e.}\quad \frac{-2}{1} = \frac{-2 \times -3}{1 \times -3} = \frac{6}{-3}$$

On the other hand, we can change $\frac{6}{-3}$ to the equal rational number $\frac{-2}{1}$ if we <u>divide</u> both its numerator and denominator by -3.

$$\text{i.e.}\quad \frac{6}{-3} = \frac{6 \div -3}{-3 \div -3} = \frac{-2}{1}$$

In general, if the numerator and denominator of any rational number are multiplied or divided by the <u>same</u> integer (not zero), an equal rational number is obtained.

Fill in the $\bigcirc$ and $\square$ to make each of the following statements true.

1. $\dfrac{1}{2} \;\overset{\times 3}{\underset{\times 3}{=}}\; \dfrac{3}{\boxed{6}}$

2. $\dfrac{-14}{-8} \;\overset{\div -2}{\underset{\div -2}{=}}\; \dfrac{\bigcirc}{\square}$

3. $\dfrac{-5}{10} \;\overset{\div 5}{\underset{\div 5}{=}}\; \dfrac{\bigcirc}{\square}$

4. $\dfrac{3}{-4} \;\overset{\times -1}{\underset{\times -1}{=}}\; \dfrac{\bigcirc}{\square}$

5. $\dfrac{-3}{-7} \;\overset{\times -2}{\underset{\times -2}{=}}\; \dfrac{\bigcirc}{\square}$

6. $\dfrac{12}{15} \;\overset{\div 3}{\underset{\div 3}{=}}\; \dfrac{\bigcirc}{\square}$

Give four rational numbers that are equivalent to the given number.

1. $\frac{-1}{4}$ $\frac{-2}{8}$, $\frac{-3}{12}$, ____, ____

2. $\frac{2}{3}$ $\frac{4}{6}$, ____, ____, ____

3. $\frac{4}{3}$ $\frac{8}{6}$, ____, ____, ____

4. $\frac{-15}{60}$ $\frac{-5}{20}$, ____, ____, ____

Fill in the numbers that should occupy the spaces in each of the following.

1. $\frac{10}{12} = \frac{-5}{-6}$ 2. $\frac{-16}{-25} = \frac{\quad}{50}$ 3. $\frac{-2}{7} = \frac{\quad}{21}$

4. $\frac{-4}{5} = \frac{\quad}{35}$ 5. $\frac{-18}{-33} = \frac{\quad}{11}$ 6. $\frac{3}{5} = \frac{-6}{\quad}$

PRINCIPLE OF EQUAL FRACTIONS

For any rational number $\frac{a}{b}$,

(i) $\frac{a}{b} = \frac{ax}{bx}$

(ii) $\frac{a}{b} = \frac{a \div x}{b \div x}$

where x is a non-zero integer.

We can use part (i) of the Principle of Equal Fractions to express a rational number in a form with a <u>larger</u> denominator. For example, the rational number $\frac{3}{5}$ can be expressed with a denominator of 30 by multiplying each of its terms by 6.

$$\frac{3}{5} = \frac{3 \times 6}{5 \times 6} = \frac{18}{30}$$

Express each fraction below with the denominator as specified.

1. $\frac{-3}{7}$ in fourteenths $\underline{\frac{-6}{14}}$

2. $\frac{3}{2}$ in tenths ______

3. $\frac{5}{6}$ in twenty-fourths ______

4. 4 in thirds ______

5. $\frac{-2}{5}$ in fifteenths ______

6. $\frac{1}{2}$ in sixteenths ______

We can use part (ii) of the Principle of Equal Fractions to express a rational number in a form with a smaller denominator. For example, the rational number $\frac{-6}{8}$ can be expressed with a denominator of 4 by dividing each of its terms by 2.

$$\frac{-6}{8} = \frac{-6 \div 2}{8 \div 2} = \frac{-3}{4}$$

This new fraction is said to be in LOWER TERMS than the original fraction.

Complete each of the following in order to obtain a fraction in lower terms.

1. $\frac{-12}{24} = \frac{-6}{12}$
2. $\frac{18}{12} = \frac{\quad}{4}$
3. $\frac{18}{12} = \frac{3}{\quad}$
4. $\frac{-12}{56} = \frac{\quad}{14}$
5. $\frac{27}{3} = \frac{\quad}{1}$
6. $\frac{-24}{32} = \frac{-3}{\quad}$

D. Expressing Rational Numbers in Lowest Terms

A rational number is in LOWEST TERMS when the only numbers which will divide evenly into both the numerator and the denominator are 1 and -1. A rational number can be reduced to lowest terms by dividing its numerator and denominator by their HIGHEST COMMON FACTOR (H.C.F.), which is the largest number that will divide evenly into both the numerator and the denominator.

EXAMPLE: Reduce the rational number $\frac{-21}{28}$ to lowest terms.

Solution. The H.C.F. of 21 and 28 is 7. This is the largest number which will divide evenly into both the numerator and the denominator.

$$\frac{-21}{28} = \frac{-21 \div 7}{28 \div 7} = \frac{-3}{4}$$

Therefore, $\frac{-21}{28}$ can be expressed in lowest terms as $\frac{-3}{4}$.

The H.C.F. of the terms of a rational number can be determined by a trial and error approach, but it can also be found in a systematic manner by going through the following steps.

Step 1: Write the numerator and denominator of the fraction as the product of prime numbers. A PRIME NUMBER is a natural number which can be divided evenly only by itself and the number 1. For example, some prime numbers are 2, 3, 5, 7, 11, 13, and so on. The number 1 is not considered to be a prime number.

<u>Step 2</u>: Compare the prime factors of the numerator and the denominator. The factors which are common to <u>both</u> terms belong to the H.C.F. If a factor occurs more than once in both terms, it will occur in the H.C.F. the greatest number of times it occurs in <u>both</u> terms. The product of these prime factors that you have picked out will be the H.C.F.

<u>Step 3</u>: To express the given fraction in lowest terms, divide the numerator and denominator by the H.C.F.

EXAMPLE: Express the fraction $\frac{-72}{90}$ in lowest terms.

<u>Solution</u>

<u>Step 1</u>: Write 72 and 90 as the product of primes.

$$72 = 8 \times 9 = \underline{2 \times 2 \times 2 \times 3 \times 3}$$

$$90 = 30 \times 3 = 10 \times 3 \times 3 = \underline{2 \times 5 \times 3 \times 3}$$

EVERY FACTOR <u>MUST</u> BE A PRIME NUMBER.

<u>Step 2</u>: Compare the prime factors and pick out those that are common to both numbers.

$$72 = 2 \times 2 \times 2 \times 3 \times 3$$

$$90 = 2 \times 5 \times 3 \times 3$$

Since the factor 2 appears <u>once</u> in both numbers, it must belong to the H.C.F.

Since the factor 3 appears <u>twice</u> in both numbers, it must appear twice in the H.C.F.

Thus, $\text{H.C.F.} = 2 \times 3 \times 3 = 18$

<u>Step 3</u>: Divide each term of the fraction $\frac{-72}{90}$ by the H.C.F. 18.

$$\frac{-72}{90} = \frac{-72 \div 18}{90 \div 18} = \frac{-4}{5}$$

Self-correcting Exercise #2

Answers may be found on page 52 of this lesson.

1. Write each number as the product of primes (i.e. each factor must be divisible by only itself and 1).

   (a) 15 = ______________ (d) 63 = ______________

   (b) 30 = ______________ (e) 54 = ______________

   (c) 32 = ______________ (f) 105 = ______________

2. Fill in the blanks in the chart. Write each pair of numbers as the product of primes and circle any common factors. Then, find the H.C.F.

	Number Pair	Product of Primes	H.C.F.
(a)	48, 168	$48 = 2 \times 2 \times 2 \times 2 \times 3$ $168 = 2 \times 2 \times 2 \times 3 \times 7$	$2 \times 2 \times 2 \times 3$ $= \underline{24}$
(b)	30, 42	30 = ______________ 42 = ______________	______________ = ______
(c)	24, 96	24 = ______________ 96 = ______________	______________ = ______
(d)	66, 165	66 = ______________ 165 = ______________	______________ = ______

3. Use your results from question #2 to reduce the following fractions to lowest terms.

   (a) $\frac{-48}{168} = \frac{-48 \div 24}{168 \div 24} = \frac{-2}{7}$ (c) $\frac{24}{96} =$

   (b) $\frac{30}{42} =$ (d) $\frac{-66}{165} =$

4. State the H.C.F. of the numerator and denominator of each fraction. Then reduce the fraction to <u>lowest</u> terms.

(a) $\frac{-75}{225}$

H.C.F. is _____.

$\frac{-75}{225}$ = _____

(b) $\frac{70}{105}$

H.C.F. is _____.

$\frac{70}{105}$ = _____

(c) $\frac{60}{108}$

H.C.F. is _____.

$\frac{60}{108}$ = _____

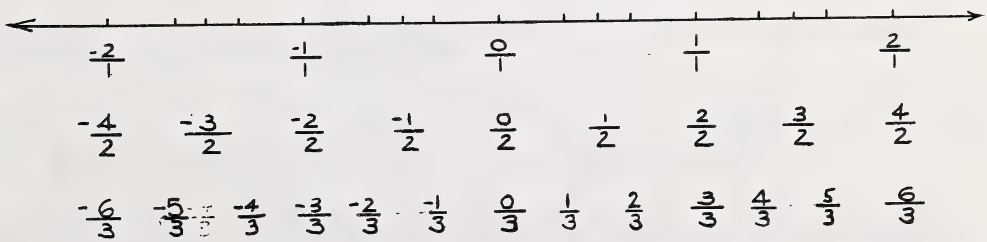
(d) $\frac{-40}{80}$

H.C.F. is _____.

$\frac{-40}{80}$ = _____

E. <u>The Rational Number Line</u>

The set of rational numbers may be associated with points on a number line. We can do this by using all the points we located on the integer line plus some additional points that fall between the integers. Thus, some points corresponding to rational numbers can be found by dividing the intervals between 0 and 1, 0 and -1, 1 and 2, -1 and -2, and so on, into halves, thirds, quarters, fifths etc. If this process of division were carried on indefinitely, every rational number could be located on the number line.



On the number line above, we have located only a few of the rational numbers which fall between -2 and 2 inclusive. Since the process of dividing this interval could go on forever, we could locate an infinite number of rational numbers between -2 and 2. In fact, between any two points on the number line, there are an infinite number of points which represent rational numbers. We describe this situation mathematically by saying that the rational numbers are DENSE.

On the other hand, the set of natural numbers and the set of integers do not have the density property. These sets are frequently referred to as being DISCRETE SETS. This means that between any two natural numbers, there is not always another natural number. A similar situation exists for integers. For example, -5 and -6 are two consecutive integers and there is no integer between them.

How many integers are there between 3 and 5? ______

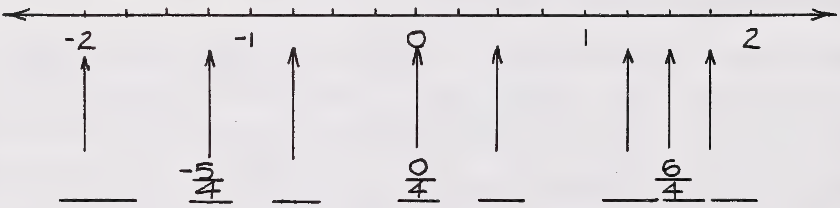
Between -1 and -4? ______ Between 2 and 3? none

Between -8 and -9? ______

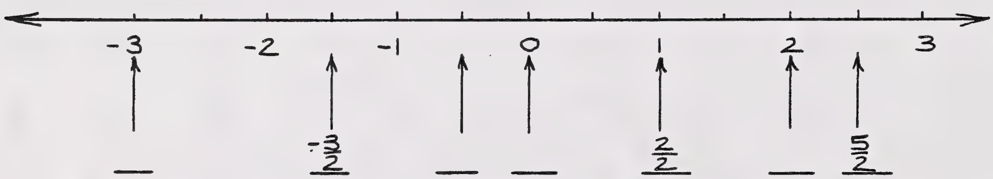
How many rational numbers are there between 0 and 1? infinite number Between -1 and -2? ______

Between $\frac{1}{3}$ and $\frac{2}{3}$? ______.

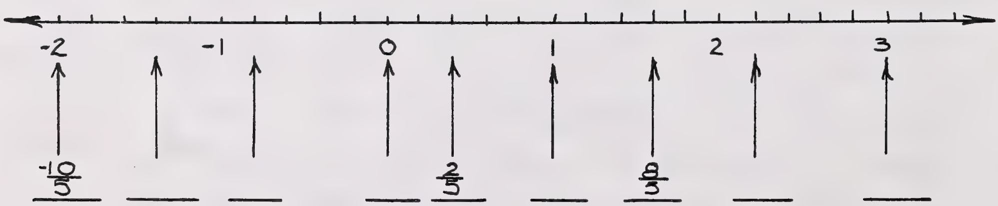
Each unit segment on the number line below has been divided into quarters. In each blank, write a rational number with a denominator of 4 that corresponds to the indicated point.



The next number line has been divided into halves. Label each indicated point with a rational number having a denominator of 2.



The last number line has been divided into fifths. Label each indicated point with a rational number having a denominator of 5.



Any positive rational number can be written in the form $\frac{a}{b}$ where a and b are natural numbers. If $a < b$ (i.e. the numerator is less than the denominator), the point corresponding to this number lies between 0 and 1 on the number line. For example, the point corresponding to $\frac{2}{5}$ lies between 0 and 1 since $\frac{2}{5}$ is a positive number and $2 < 5$. Put a check mark beside each rational number below that lies between 0 and 1.

1. $\frac{5}{8}$ ✓ 2. $\frac{8}{7}$ ____ 3. $\frac{-1}{2}$ ____

4. $\frac{4}{8}$ ____ 5. $\frac{16}{17}$ ____ 6. $\frac{-6}{13}$ ____

Any negative rational number can be written in the form $\frac{-a}{b}$ where a and b are natural numbers. If $a < b$ (i.e. the absolute value of the numerator is less than the denominator), the point corresponding to this number lies between 0 and -1 on the number line. For example, the point corresponding to $\frac{-3}{8}$ lies between 0 and -1 since $\frac{-3}{8}$ is a negative number and $3 < 8$. Put a check mark beside each rational number below that lies between 0 and -1.

1. $\frac{-3}{4}$ ✓ 2. $\frac{-16}{12}$ ____ 3. $\frac{10}{3}$ ____

4. $\frac{-1}{6}$ ____ 5. $\frac{-7}{9}$ ____ 6. $\frac{-14}{2}$ ____

F. Ordering of the Rational Numbers

In Lesson 3, you learned that if a and b are two integers and a lies to the right of b on the number line, then a is greater than b ($a > b$). If a lies to the left of b, then a is less than b ($a < b$).

Rational numbers are ordered in a similar fashion. If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers:

$\frac{a}{b} > \frac{c}{d}$ if $\frac{a}{b}$ lies to the right of $\frac{c}{d}$.

$\frac{a}{b} < \frac{c}{d}$ if $\frac{a}{b}$ lies to the left of $\frac{c}{d}$.

Use the number line on page 13 of this lesson to help you answer the following questions. Fill in the first blank with the symbol ">" or "<" and the second blank with the word "right" or "left".

1. $\frac{-3}{2}$ $\underline{\ <\ }$ $\frac{-1}{2}$ since $\frac{-3}{2}$ lies to the $\underline{\text{left}}$ of $\frac{-1}{2}$.

2. $\frac{3}{2}$ ______ $\frac{1}{2}$ since $\frac{3}{2}$ lies to the ________ of $\frac{1}{2}$.

3. $\frac{-1}{3}$ ______ $\frac{1}{2}$ since $\frac{-1}{3}$ lies to the ________ of $\frac{1}{2}$.

4. $\frac{-1}{3}$ ______ $\frac{-2}{3}$ since $\frac{-1}{3}$ lies to the ________ of $\frac{-2}{3}$.

5. $\frac{5}{3}$ ______ $\frac{1}{2}$ since $\frac{5}{3}$ lies to the ________ of $\frac{1}{2}$.

6. $\frac{-3}{2}$ ______ $\frac{-1}{3}$ since $\frac{-3}{2}$ lies to the ________ of $\frac{-1}{3}$.

7. $\frac{1}{2}$ ______ $\frac{-1}{3}$ since $\frac{1}{2}$ lies to the ________ of $\frac{-1}{3}$.

There is another method that you can use for ordering two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$. If the numbers are expressed with positive denominators, you can find the cross products of their terms and apply the following rule:

If $ad > bc$, then $\frac{a}{b} > \frac{c}{d}$.

If $ad < bc$, then $\frac{a}{b} < \frac{c}{d}$.

EXAMPLES:

$\frac{2}{3} < \frac{3}{4}$ since $2 \times 4 = 8$, $3 \times 3 = 9$, and $8 < 9$.

$\frac{-1}{5} > \frac{-5}{11}$ since $-1 \times 11 = -11$, $5 \times -5 = -25$, and $-11 > -25$.

(-11 IS GREATER SINCE IT LIES TO THE RIGHT OF -25)

Thus, you must go through the following steps when ordering two rational numbers.

1. Make sure the fractions have positive denominators.

2. Find the product of the numerator of the first fraction and the denominator of the second.

3. Find the product of the denominator of the first fraction and the numerator of the second.

4. Compare the two cross products.
   - The first fraction is the greater one if the first cross product is greater than the second.
   - The first fraction is the lesser one if the first cross product is less than the second.

Self-correcting Exercise #3

Answers may be found on page 54 of this lesson.

1. Fill in the blanks in the chart below in order to determine if the first fraction is greater than or less than the second.

	$\frac{a}{b}$, $\frac{c}{d}$	Cross Products: ad	Cross Products: bc	Comparison of Cross Products	Ordering of the Fractions
(a)	$\frac{-4}{3}$, $\frac{-5}{2}$	$-4 \times 2 = -8$	$3 \times -5 = -15$	$-8 > -15$ -8 LIES TO THE RIGHT OF −15.	$\frac{-4}{3} > \frac{-5}{2}$
(b)	$\frac{3}{14}$, $\frac{5}{26}$	__ × __ = __	__ × __ = __	____	____
(c)	$\frac{2}{3}$, $\frac{-2}{3}$	__ × __ = __	__ × __ = __	____	____
(d)	$\frac{-5}{6}$, $\frac{-3}{4}$	__ × __ = __	= __ × __ = __	____	____
(e)	$\frac{7}{19}$, $\frac{2}{5}$	__ × __ = __	__ × __ = __	____	____

2. Insert the symbol "<", ">", or "=" between each of the following pairs of rational numbers.

(a) $\frac{4}{5}$ _____ $\frac{5}{6}$

(d) $\frac{3}{6}$ _____ $\frac{-1}{2}$

(b) $\frac{-3}{2}$ _____ $\frac{11}{8}$

(e) $\frac{10}{4}$ _____ $\frac{5}{2}$

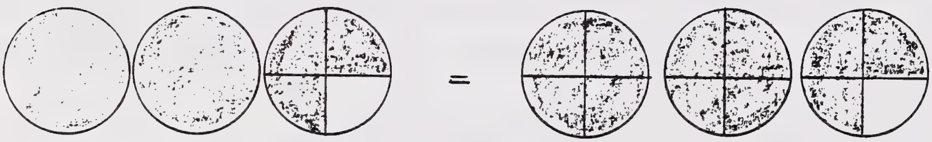
(c) $\frac{2}{3}$ _____ $\frac{10}{15}$

(f) $\frac{-11}{12}$ _____ $\frac{-5}{6}$

G. Mixed Numbers

A positive MIXED NUMBER expresses the sum of a natural number and a fraction. It tells us how many "wholes" and "parts of a whole" are represented by a particular positive rational number.

For example, $2\frac{3}{4}$ is a positive mixed number. It represents the sum of two "wholes" and three quarters of a whole.



$$2\frac{3}{4} = \frac{11}{4}$$

Note that the mixed number $2\frac{3}{4}$ is equivalent to the rational number $\frac{11}{4}$. We can change this mixed number to its rational form by using the following procedure.

1. Multiply the whole number part of the mixed number by the denominator of the fractional part.

$$2 \times 4 = 8$$

2. To the result, add the numerator of the fractional part.

$$8 + 3 = 11$$

11 is the numerator of the equivalent rational number.

3. The denominator of the equivalent rational number is the same as the denominator of the fractional part of the mixed number (i.e. the denominator is 4).

$$2\frac{3}{4} = \frac{(2 \times 4) + 3}{4}$$

We can shorten the work by doing the middle step mentally.

$$2\,\frac{3}{4} = \text{Think: } [(2 \times 4) + 3] \text{ quarters} = \frac{11}{4}$$

Write each of the following positive mixed numbers in rational form by filling in the blanks.

1. $3\frac{2}{5}$ = Think: [(3 × __) + 2] fifths = $\frac{\quad}{5}$

2. $7\frac{1}{8}$ = Think: [(__ × 8) + 1] eighths = $\frac{\quad}{8}$

3. $5\frac{4}{7}$ = Think: [(__ × __) + 4] sevenths = $\frac{\quad}{7}$

4. $2\frac{5}{9}$ = Think: [(__ × __) + __] ninths = —

5. $7\frac{2}{3}$ = Think: ________________ = —

Similarly, negative mixed numbers represent negative rational numbers. For example, $-2\frac{1}{3}$ is a negative mixed number. We can change it to a rational number by using the same procedure we used for positive numbers, but attaching a minus sign to the numerator of the rational number.

NUMERATOR IS NEGATIVE.

$$-2\frac{1}{3} = \text{Think: } -[(2 \times 3) + 1] \text{ thirds} = \frac{-7}{3}$$

NUMERATOR IS NEGATIVE.

Write each of the following negative mixed numbers in rational form by filling in the blanks.

1. $-4\frac{3}{5}$ = Think: $-[(4 \times \underline{\quad}) + 3]$ fifths = $\frac{\quad}{5}$

2. $-5\frac{1}{2}$ = Think: $-[(\underline{\quad} \times \underline{\quad}) + 1]$ halves = $\frac{\quad}{2}$

3. $-1\frac{9}{10}$ = Think: ______________________ = $\frac{\quad}{10}$

Any positive rational number in which the numerator is larger than the denominator can be changed to a mixed number by dividing the denominator into the numerator. The quotient gives you the whole number part of the mixed number. The remainder is placed over the divisor to give you the fractional part of the mixed number.

For example, we can change the positive rational number $\frac{9}{2}$ to a mixed number by dividing 2 into 9.

$$9 \div 2 = 4, \text{ remainder } 1$$

The quotient, 4, tells us the number of "wholes" there are in $\frac{9}{2}$.

To find the fractional part of the mixed number, we place the remainder, 1, over the divisor, 2.

$$9 \div 2 = 4\frac{1}{2} \begin{array}{l} \leftarrow \text{REMAINDER} \\ \leftarrow \text{DIVISOR} \end{array}$$

Similarly, negative rational numbers in which the absolute value of the numerator is greater than the denominator may be changed to negative mixed numbers. For example, $\frac{-19}{4}$ is a negative rational number. We can change it to a mixed number by using the same procedure we used for positive numbers, but attaching a minus sign to the result.

$\frac{-19}{4} = -4\frac{3}{4}$

NEGATIVE RATIONAL NUMBER   NEGATIVE MIXED NUMBER

Think: $19 \div 4 = 4$, remainder 3

$= 4\frac{3}{4}$

Change each rational number to a mixed number by filling in the blanks in the following chart.

	Rational Number	Required Division	Mixed Number
1.	$\frac{-21}{8}$	$21 \div 8 =$ ___, remainder ___	$-2\frac{\ }{8}$
2.	$\frac{35}{6}$	$35 \div 6 =$ ___, remainder ___	______
3.	$\frac{11}{10}$	$11 \div 10 =$ ___, remainder ___	______
4.	$\frac{-17}{2}$	$17 \div 2 =$ ___, remainder ___	______
5.	$\frac{63}{5}$	$63 \div 5 =$ ___, remainder ___	______

Self-correcting Exercise #4

Answers may be found on page 55 of this lesson.

1. Change each rational number to a mixed number.

(a) $\frac{13}{2} =$ ______ (b) $\frac{-35}{3} =$ ______ (c) $\frac{17}{6} =$ ______ (d) $\frac{-60}{7} =$ ______

2. Change each mixed number to a rational number.

(a) $5\frac{3}{8} = \frac{\ }{8}$ (b) $-1\frac{5}{7} =$ ___ (c) $-3\frac{2}{3} =$ ___ (d) $6\frac{2}{5} =$ ___

3. Write each rational number in lowest terms and then represent it by a mixed number.

(a) $\frac{45}{36} = \frac{45 \div 9}{36 \div 9} = \frac{5}{4} = 1\frac{1}{4}$ (b) $\frac{-18}{4} =$

(c) $\frac{-96}{56} =$ (d) $\frac{80}{15} =$

EXERCISE - The Set of Rational Numbers

1. Fill in the blanks.

(a) With respect to the fraction $\frac{3}{5}$, the ________________ of the fraction is 3 and the ______________ is 5. 3 and 5 are called the ______________ of the fraction.

(b) If $\frac{m}{n}$ and $\frac{p}{q}$ are equal rational numbers, then mq = ___________.

(c) A ___________ number is only divisible by itself and one.

(d) All negative rational numbers are ___________ than all positive rational numbers.

(e) If the rational number $\frac{c}{d}$ is less than the rational number $\frac{e}{f}$, (where d and f are positive) then cf is ________ than _________.

(f) Rational numbers are said to be __________ because between any pair of rational numbers there are an infinite number of rational numbers.

(g) The prime factors of 70 are ___, ___, and ___.

(h) An equal rational number is obtained if the numerator and denominator of a rational number are __________________ or __________________ by the same non-zero integer.

(i) The rational numbers $\frac{-6}{-9}$ and $\frac{6}{9}$ are __________________ than zero while the rational numbers $\frac{-6}{9}$ and $\frac{6}{-9}$ are __________________ than zero.

(j) $\frac{-13}{0}$ is not a rational number because division by ______________ is impossible.

(k) The rational number $\frac{16}{24}$ can be written in lowest terms by _______________ its numerator and denominator by ___________.

(l) $\frac{-6}{3}$ is a rational number, but it is also an _______________ because the denominator divides evenly into the numerator.

(m) $\frac{-5}{8} < \frac{-9}{15}$ since $-5 \times$ __ $< 8 \times$ __.

(n) The rational number $\frac{31}{7}$ can be written as the ______________ number $4\frac{3}{7}$.

(o) The largest number which will divide evenly into two given number is called the ____________ ____________ factor of those numbers.

(p) One rational number is greater than another if it lies to the ____________ of the other on the number line.

(q) The set of integers is a ______________ of the set of rational numbers.

2. Give a number in rational form that is equivalent to each of the following integers.

(a) 5 $\frac{35}{7}$ (b) -3 = _____ (c) 2 = _____

(d) 0 = _____ (e) -11 = _____ (f) 6 = _____

3. Write each number as the product of prime factors. Then state the highest common factor of each pair.

(a) 64 and 40

64 = 2×2×2×2×2×2

40 = 2×2×2×5

H.C.F. = 2×2×2

= 8

(b) 490 and 245

490 = __________

245 = __________

FACTORS YOU LIST HERE MUST BE PRIME NUMBERS.

H.C.F. = __________

= _____

(c) 144 and 108

144 = __________

108 = __________

H.C.F. = __________

= _____

(d) 75 and 135

75 = __________

135 = __________

H.C.F. = __________

= _____

4. Reduce each fraction to lowest terms by filling in the blanks in the following chart.

A PRIME FACTOR IS DIVISIBLE BY ONLY ITSELF AND 1.

Fraction	Step 1: Find prime factors of both terms.	Step 2: Find H.C.F.	Step 3: Express fraction in lowest terms.
$\frac{-9}{15}$	$9 = 3 \times 3$ $15 = 3 \times 5$	H.C.F. = 3	$\frac{-9 \div 3}{15 \div 3} = \frac{-3}{5}$
$\frac{30}{50}$			
$\frac{-75}{225}$			
$\frac{48}{168}$			
$\frac{-70}{105}$			

5. Circle all the rational forms which are equivalent to each of the given rational numbers.

	Rational Number	Circle Equivalent Forms
(a)	$\frac{2}{8}$	$\frac{-2}{-8}$ $\frac{2}{-8}$, $\frac{-2}{8}$, $\frac{-1}{-4}$, $\frac{8}{24}$, $\frac{1}{4}$, $\frac{-20}{80}$
(b)	$\frac{-4}{5}$	$\frac{4}{5}$, $\frac{4}{-5}$, $\frac{-4}{-5}$, $\frac{12}{15}$, $\frac{-8}{10}$, $\frac{-20}{25}$, $\frac{-12}{10}$
(c)	-12	$\frac{-36}{3}$, $\frac{36}{-3}$, $\frac{36}{3}$, $\frac{-12}{-1}$, $\frac{12}{-1}$, $\frac{-12}{1}$, $\frac{60}{-5}$

6. Find the cross products of the terms in each pair of fractions. Then order the fractions by inserting the symbol <, >, or = between them.

$\frac{a}{b}$ $\frac{c}{d}$	Cross-Products		Ordering of Fractions
	ad	bc	
$\frac{1}{3}$, $\frac{-2}{3}$	$1 \times 3 = 3$	$3 \times -2 = -6$	$\frac{1}{3} > \frac{-2}{3}$
$\frac{-2}{5}$, $\frac{-16}{15}$			
$\frac{9}{25}$, $\frac{11}{30}$			
$\frac{-8}{3}$, $\frac{-48}{18}$			
$\frac{-21}{36}$, $\frac{-7}{12}$			

See pages 10 and 11.

7. Express each rational number in lowest terms. Then, rewrite the rational number as a mixed number.

(a) $\frac{-234}{126} = \frac{-234 \div 18}{126 \div 18} = \frac{-13}{7} = -1\frac{6}{7}$

(b) $\frac{84}{36} =$

(c) $\frac{-54}{16} =$

(d) $\frac{-90}{72} =$

8. Change each rational number to a mixed number in order to determine the two consecutive integers it would lie between on the number line.

Rational Number	Mixed Number	Position on Number Line
$\frac{-31}{7}$	$-4\frac{3}{7}$	Lies between -4 and -5.
$\frac{13}{2}$	______	______________
$\frac{-65}{12}$	______	______________
$\frac{14}{9}$	______	______________
$\frac{-19}{8}$	______	______________
$\frac{99}{41}$	______	______________

9. (a) What two consecutive integers does $\frac{15}{17}$ lie between? ______________

(b) What two consecutive integers does $\frac{-7}{8}$ lie between? ______________

10. Insert the symbol "<", ">" or "=" between each of the following pairs of rational numbers.

(a) $\frac{-6}{11}$ ______ $\frac{-17}{32}$

(b) $\frac{-5}{9}$ ______ $\frac{-55}{99}$

(c) $\frac{7}{18}$ ______ $\frac{12}{32}$

(d) $\frac{9}{12}$ ______ $\frac{15}{20}$

Topic Two: Operating With Rational Numbers

A. Multiplying Rational Numbers

We can use what we know about multiplying integers to help us multiply fractions. For example, in Set I, we know that

$$2 \times (-3) = -6$$

The integers 2 and -3 could be written in rational form as $\frac{4}{2}$ and $\frac{-12}{4}$. The product of these two rational numbers should also equal -6. Note that if we find the product of the numerators of the two rational numbers and divide the result by the product of the denominators, we will arrive at a rational number that is equivalent to -6.

$$\text{i.e.} \quad \frac{4}{2} \times \frac{-12}{4} = \frac{4 \times -12}{2 \times 4} = \frac{-48}{8} = -6$$

In view of the above example, what do you think the following products should be?

$\frac{3}{5} \times \frac{2}{7} = \frac{6}{35}$ $\qquad$ $\frac{-1}{4} \times \frac{-3}{2} =$ ____

$\frac{-1}{8} \times \frac{7}{9} =$ ____ $\qquad$ $\frac{-3}{4} \times \frac{1}{2} =$ ____

MULTIPLICATION - TWO RATIONAL NUMBERS

In order to find the product of two rational numbers, multiply the numerators and divide this result by the product of the denominators.

$$\text{i.e.} \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Use the multiplication rule above to find each of the following products. Then, express your answer in lowest terms by dividing the numerator and denominator by the same amount.

1. $\frac{3}{4} \times \frac{4}{5} = \frac{3 \times 4}{4 \times 5} = \frac{12}{20} = \frac{3}{5}$

2. $\frac{-5}{8} \times \frac{4}{10} = \frac{-5 \times 4}{8 \times 10} = \frac{\quad}{80} = \frac{\quad}{4}$

3. $\frac{-2}{5} \times \frac{-3}{16} = \frac{}{5 \times 16} = \frac{}{80} = \frac{}{}$

4. $\frac{5}{9} \times \frac{-18}{25} = \frac{5 \times -18}{} = \frac{-90}{} = \frac{-2}{}$

There is short-cut method for finding the product of two rational numbers that always results in a rational number that is expressed in lowest terms. Using this method, the numerator and denominator of the product can be simplified before the multiplication is completed.

EXAMPLE: Find the product of $\frac{-3}{4}$ and $\frac{-8}{9}$.

<u>Solution</u>

$$\frac{-3}{4} \times \frac{-8}{9} = \frac{-3 \times -8}{4 \times 9}$$

At this point, the numerator and denominator can both be divided by the common factor 3 (i.e. 3 divides evenly into -3 in the numerator and 9 in the denominator).

$$= \frac{\overset{-1}{-\cancel{3}} \times -8}{4 \times \underset{3}{\cancel{9}}}$$

The numeral -1 is written above the factor -3 in the numerator to indicate that -3 ÷ 3 = -1. The numeral 3 is placed below the factor 9 in the denominator to indicate that 9 ÷ 3 = 3.

Also, the numerator and denominator can both be divided by the common factor 4 (i.e. 4 divides evenly into -8 in the numerator and 4 in the denominator).

$$= \frac{\overset{-1}{-\cancel{3}} \times \overset{-2}{-\cancel{8}}}{\underset{1}{\cancel{4}} \times \underset{3}{\cancel{9}}}$$

The numeral -2 is placed above the factor -8 in the numerator to indicate that -8 ÷ 4 = -2. The numeral 1 is placed below the factor 4 in the denominator to indicate that 4 ÷ 4 = 1.

Now that the fraction has been expressed in lowest terms, the multiplication can be completed.

$$= \frac{-1 \times -2}{1 \times 3}$$

$$= \frac{2}{3}$$

Self-correcting Exercise #5

Answers may be found on page 55 of this lesson.

1. Simplify each product by dividing the numerator and denominator by common factors. (Do not complete the multiplication.) Part (a) has been done for you.

(a) $\frac{2}{3} \times \frac{15}{16} = \frac{\overset{1}{\cancel{2}} \times \overset{5}{\cancel{15}}}{\underset{1}{\cancel{3}} \times \underset{8}{\cancel{16}}}$

(b) $\frac{3}{8} \times \frac{4}{9} = \frac{3 \times 4}{8 \times 9}$

(c) $\frac{3}{14} \times \frac{7}{8} = \frac{3 \times 7}{14 \times 8}$

(d) $\frac{-7}{8} \times \frac{4}{21} = \frac{-7 \times 4}{8 \times 21}$

(e) $\frac{-4}{15} \times \frac{-5}{8} = \frac{-4 \times -5}{15 \times 8}$

(f) $\frac{21}{25} \times \frac{-5}{7} = \frac{21 \times -5}{25 \times 7}$

2. Find the following products using the short-cut method. All answers must be in lowest terms. Where possible, express answers as mixed numbers.

(a) $\frac{8}{3} \times \frac{3}{5} = \frac{8 \times \overset{1}{\cancel{3}}}{\underset{1}{\cancel{3}} \times 5} =$

(b) $\frac{-15}{4} \times \frac{16}{3} =$

(c) $\frac{52}{17} \times \frac{34}{39} =$

3. Write each mixed number as a rational number. Then find the product. Answers must be expressed in lowest terms and, where possible, as mixed numbers.

(a) $-6\frac{1}{4} \times 2\frac{4}{5} = \frac{-25}{4} \times \frac{14}{5} =$

(b) $2\frac{5}{9} \times 2\frac{1}{4} =$

B. Dividing Rational Numbers

Any division question can be thought of as a multiplication question since multiplication and division are inverse operations. For example, the division question $8 \div 2 = \boxed{}$ is equivalent to the multiplication question $2 \times \boxed{} = 8$. In both cases, the missing number is 4. Since multiplication and division are related in this manner, we can use our knowledge of multiplying rational numbers to help us develop rules for dividing rational numbers.

Suppose we wished to find the quotient:

$$\frac{1}{2} \div \frac{1}{3} = \boxed{}$$

We could change this division question to a multiplication question by asking ourselves, "What number must we multiply $\frac{1}{3}$ by so that it will equal $\frac{1}{2}$?" Thus, the equivalent multiplication question would be:

$$\frac{1}{2} = \frac{1}{3} \times \boxed{}$$

From our rules for multiplying rational numbers, we know that:

$$\frac{1}{2} = \frac{1}{3} \times \boxed{\frac{3}{2}}$$

Changing back to a division question, we can say that:

$$\frac{1}{2} \div \frac{1}{3} = \boxed{\frac{3}{2}}$$

Note that the same result can be obtained if we change the operation to multiplication and interchange the numerator and denominator of the divisor.

i.e. $$\underbrace{\frac{1}{2} \div \frac{1}{3}}_{\text{DIVIDE BY } \frac{1}{3}} = \underbrace{\frac{1}{2} \times \frac{3}{1}}_{\text{MULTIPLY BY } \frac{3}{1}} = \frac{1 \times 3}{2 \times 1} = \frac{3}{2}$$

When we interchange the numerator and denominator of a fraction, we obtain the RECIPROCAL of that fraction.

EXAMPLES:

1. The reciprocal of $\frac{-2}{3}$ is $\frac{3}{-2}$.

2. The reciprocal of 5 is $\frac{1}{5}$.

3. The reciprocal of $\frac{4}{5}$ is ______.

4. The reciprocal of $\frac{-7}{3}$ is ____.

} Fill in the blanks.

DIVISION - TWO RATIONAL NUMBERS
In order to divide one rational number by another, change the operation to multiplication and give the reciprocal of the divisor. Then, apply the rule for multiplying rational numbers.
i.e. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

Change operation to multiplication. *Give reciprocal of divisor.*

Thus, all division questions can be changed to multiplication questions and the rules for multiplying rational numbers can then be applied.

EXAMPLE: $\frac{7}{8} \div \frac{-21}{40} = \boxed{}$

Solution

First, change the operation to multiplication, and give the reciprocal of the divisor.

$$\frac{7}{8} \div \frac{-21}{40} = \frac{7}{8} \times \frac{40}{-21}$$

multiply reciprocal of divisor

Then, proceed as in multiplication.

$$= \frac{\overset{1}{\cancel{7}} \times \overset{5}{\cancel{40}}}{\underset{1}{\cancel{8}} \times -\underset{-3}{\cancel{21}}}$$

$$= \frac{1 \times 5}{1 \times -3}$$

$$= \frac{5}{-3}$$

$$= -1\frac{2}{3}$$

Self-correcting Exercise #6

Answers may be found on page 56 of this lesson.

1. Name the reciprocal of each rational number.

(a) $\frac{8}{3}$ ________ (b) 2 ________ (c) $\frac{-5}{12}$ ________

(d) -3 ________ (e) $\frac{2}{-5}$ ________ (f) $\frac{1}{4}$ ________

2. Rewrite each division question as a multiplication question. (Do not complete the multiplication.)

(a) $\frac{2}{3} \div \frac{1}{4} =$ $\frac{2}{3} \times \frac{4}{1}$ (d) $\frac{6}{5} \div \frac{-3}{2} =$ ________

(b) $5 \div \frac{3}{7} =$ ________ (e) $\frac{1}{8} \div \frac{1}{6} =$ ________

(c) $\frac{2}{3} \div 8 =$ ________ (f) $\frac{-1}{4} \div -5 =$ ________

3. Change each division question to a multiplication question and then find the product. Express answers in lowest terms and, where possible, as mixed numbers.

(a) $\frac{2}{3} \div \frac{-1}{6} =$

(b) $\frac{11}{12} \div \frac{3}{4} =$

(c) $\frac{7}{4} \div \frac{3}{16} =$

(d) $\frac{-2}{3} \div 8 =$

4. Write each mixed number as a rational number and then divide. Express answers in lowest terms, and where possible, as mixed numbers.

(a) $3\frac{3}{5} \div \frac{9}{10} =$

(b) $-7 \div 4\frac{1}{5} =$

(c) $4\frac{1}{8} \div 11 =$

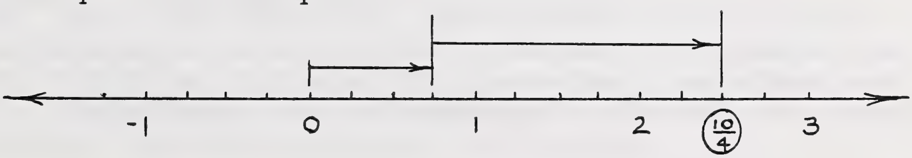
C. Adding and Subtracting Rational Numbers With Like Denominators

We can use a number line to help us add and subtract rational numbers with like denominators.

EXAMPLE 1: $\frac{3}{4} + \frac{7}{4} = \boxed{}$

Solution

The rational number $\frac{3}{4}$ can be represented on the number line by drawing a ray that begins at 0 and ends at $\frac{3}{4}$. Then, we can represent the operation of adding $\frac{7}{4}$ by drawing a ray that begins at $\frac{3}{4}$ and covers $\frac{7}{4}$ spaces to the right.



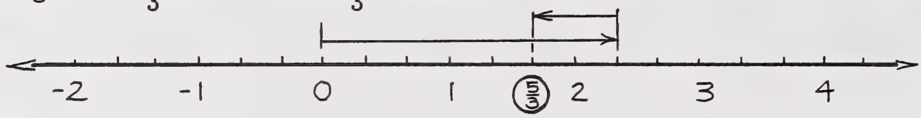
From the number line, we can see that:

$$\frac{3}{4} + \frac{7}{4} = \frac{10}{4}$$

EXAMPLE 2: $\frac{7}{3} - \frac{2}{3} = \square$

Solution

The rational number $\frac{7}{3}$ can be represented on the number line by drawing a ray that begins at 0 and ends at $\frac{7}{3}$. Then, in order to represent the operation of subtracting $\frac{2}{3}$, we can draw a ray that begins at $\frac{7}{3}$ and covers $\frac{2}{3}$ spaces to the left.



From the number line, we can see that:

$$\frac{7}{3} - \frac{2}{3} = \frac{5}{3}$$

On a piece of scrap paper, use number lines to help you find the following sums and differences. (Fill in the missing numerators.)

$\frac{3}{8} + \frac{2}{8} = \frac{\quad}{8}$ $\qquad$ $\frac{1}{6} + \frac{4}{6} = \frac{\quad}{6}$

$\frac{5}{12} - \frac{4}{12} = \frac{\quad}{12}$ $\qquad$ $\frac{4}{7} - \frac{1}{7} = \frac{\quad}{7}$

Look at the four answers above. Note that in each case, the denominator of the answer is the same as the denominators of the two fractions. The numerator of the answer can be found by adding or subtracting the numerators of the two fractions.

ADDITION AND SUBTRACTION - TWO RATIONAL NUMBERS WITH THE SAME DENOMINATOR
To add two rational numbers with the same denominator, add the numerators. Write the sum over the common denominator.
To subtract two rational numbers with the same denominator, subtract the numerators. Write the difference over the common denominator.

EXAMPLES: Write answer in simplest form.

SUM OF NUMERATORS

$$\frac{5}{8} + \frac{7}{8} = \frac{5+7}{8} = \underbrace{\frac{12}{8} = \frac{12 \div 4}{8 \div 4} = \frac{3}{2} = 1\frac{1}{2}}$$

COMMON DENOMINATOR

DIFFERENCE OF NUMERATORS

$$\frac{17}{24} - \frac{11}{24} = \frac{17-11}{24} = \frac{6}{24} = \frac{6 \div 6}{24 \div 6} = \frac{1}{4}$$

COMMON DENOMINATOR

Using the rules for adding and subtracting rational numbers, find the following sums and differences. Express answers in lowest terms and as mixed numbers, where possible.

1. $\frac{7}{9} - \frac{-5}{9} = \frac{7-(-5)}{9} = \frac{7+(+5)}{9} = \frac{12}{9} = \frac{4}{3} = 1\frac{1}{3}$

2. $\frac{7}{10} + \frac{3}{10} =$

3. $\frac{1}{9} + \frac{5}{9} - \frac{2}{9} = \frac{1+5-2}{9} =$

4. $\frac{14}{3} - \frac{18}{3} + \frac{2}{3} = \frac{14-18+2}{3} =$

D. <u>Changing Two Rational Numbers to the Same Denominator</u>

When working with two rational numbers, we are often faced with the task of changing these numbers to equivalent forms which have the same denominator. The new denominator which is chosen is called a COMMON DENOMINATOR of the two fractions.

EXAMPLE: The rational numbers $\frac{5}{8}$ and $\frac{7}{12}$ can be changed to the the common denominator 96.

$$\frac{5}{8} = \frac{5 \times 12}{8 \times 12} = \frac{60}{96}$$

$$\frac{7}{12} = \frac{7 \times 8}{12 \times 8} = \frac{56}{96}$$

Our work is often simplified, though, if we choose the smallest common denominator. The <u>smallest</u> denominator that two fractions can be changed to is called the LEAST COMMON DENOMINATOR (L.C.D.) of the two fractions.

EXAMPLE: The rational numbers $\frac{5}{8}$ and $\frac{7}{12}$ can be changed to the <u>least</u> common denominator 24. Note that 24 is the smallest number into which both 8 and 12 will divide. In order to change $\frac{5}{8}$ to twenty-fourths, we must multiply the numerator and denominator by 3. In order to change $\frac{7}{12}$ to twenty-fourths, we must multiply the numerator and denominator by 2.

$$\frac{5}{8} = \frac{5 \times 3}{8 \times 3} = \frac{15}{24}$$

$$\frac{7}{12} = \frac{7 \times 2}{12 \times 2} = \frac{14}{24}$$

Although the least common denominator of two fractions can be determined by a trial and error approach, this denominator can also be found in a systematic manner by going through the following steps.

<u>Step 1:</u> Write the denominators of the two fractions as the product of prime numbers. (Remember that a prime number is a natural number that is divisible only by itself and one.)

<u>Step 2:</u> Compare the prime factors of the two denominators. If a prime factor occurs in only one of the denominators, it must be part of the L.C.D. If a prime factor occurs in both denominators it will occur in the L.C.D. the <u>greatest</u> number of times it occurs in one of the denominators. The product of these prime numbers you have picked out will be the L.C.D.

<u>Step 3:</u> Change each fraction to the L.C.D. by using the Principle of Equal Fractions.

EXAMPLE: Change the fractions $\frac{5}{24}$ and $\frac{-3}{80}$ to their L.C.D.
<u>Solution</u>

<u>Step 1:</u> The denominator 24 can be written as $2 \times 2 \times 2 \times 3$.

The denominator 80 can be written as $2 \times 2 \times 2 \times 2 \times 5$.

Step 2:

The factor 3 appears in only one number so must belong to the L.C.D.

The factor 5 appears in only one number so must belong to the L.C.D.

The factor 2 appears in both numbers. Since it appears in one number 4 times, it must appear in the L.C.D. 4 times.

Thus, L.C.D. $= 3 \times 5 \times 2 \times 2 \times 2 \times 2 = \underline{240}$

Step 3:

We must multiply each term of $\frac{5}{24}$ by 10 and each term of $\frac{-3}{80}$ by 3 in order to change them both to 240$^{\text{ths}}$.

$$\frac{5}{24} = \frac{5 \times 10}{24 \times 10} = \frac{50}{240}$$

$$\frac{-3}{80} = \frac{-3 \times 3}{80 \times 3} = \frac{-9}{240}$$

Self-correcting Exercise #7

Answers may be found on page 57 of this lesson.

1. Fill in the blanks in the chart. Write each denominator as the product of primes and then find the L.C.D. (Pick out each prime factor the <u>greatest</u> number of times it occurs in any one denominator.)

	Fraction Pairs	Denominators Expressed as Product of Primes	L.C.D.
(a)	$\frac{3}{108}$, $\frac{5}{72}$	$108 = 2 \times 2 \times 3 \times 3 \times 3$ $72 = 2 \times 2 \times 2 \times 3 \times 3$	$3 \times 3 \times 3 \times 2 \times 2 \times 2$ = ____
(b)	$\frac{5}{6}$, $\frac{-8}{9}$	6 = ________ 9 = ______	________ = ____
(c)	$\frac{7}{24}$, $\frac{5}{18}$	24 = ________ 18 = ______	________ = ____
(d)	$\frac{7}{15}$, $\frac{10}{27}$	15 = ________ 27 = ______	________ = ____

2. In the following chart, pairs of fractions are taken to their least common denominator. Fill in the blanks that appear in the chart.

Fraction Pairs	Step 1: Find prime factors of denominators.	Step 2: Find L.C.D.	Step 3: Express fractions in terms of L.C.D.
$\frac{3}{16}$ and $\frac{-5}{36}$	$16 = 2 \times 2 \times 2 \times 2$ $36 = 2 \times 2 \times 3 \times 3$	L.C.D. $= 2 \times 2 \times 2 \times 2 \times 3 \times 3$ $= \_\_\_\_$	$\frac{3}{16} = \frac{3 \times 9}{16 \times 9} = \frac{\_\_\_}{144}$ $\frac{-5}{36} = \frac{-5 \times 4}{36 \times 4} = \frac{\_\_\_}{144}$
$\frac{-7}{60}$ and $\frac{-13}{90}$			
$\frac{3}{56}$ and $\frac{-20}{98}$			
$\frac{-5}{18}$ and $\frac{7}{24}$			

E. <u>Adding and Subtracting Rational Numbers With Unlike Denominators</u>

In the last section you learned how to change two rational numbers to their least common denominator. In order to add or subtract two rational numbers with unlike denominators, you must first change these fractions to their least common denominator. Then the rules for adding and subtracting two rational numbers with the same denominator can be applied.

EXAMPLE: Find the sum of $\frac{-7}{60}$ and $\frac{-13}{90}$.

<u>Solution</u>

Your first concern is to find the L.C.D. of $\frac{-7}{60}$ and $\frac{-13}{90}$.

(Review example on pages 36 and 37 of this lesson.)

Note that:

$$\begin{aligned} 60 &= 2 \times 2 \times 3 \times 5 \\ 90 &= 2 \times 3 \times 3 \times 5 \\ \text{L.C.D.} &= 2 \times 2 \times 3 \times 3 \times 5 \\ &= 180 \end{aligned}$$

Each fraction can be expressed in terms of the L.C.D.

$$\frac{-7}{60} = \frac{-7 \times 3}{60 \times 3} = \frac{-21}{180}$$

$$\frac{-13}{90} = \frac{-13 \times 2}{90 \times 2} = \frac{-26}{180}$$

Since the two fractions now have the same denominator, the rule for adding two rational numbers with like denominators can be applied.

$$\frac{-21}{180} + \frac{-26}{180} = \frac{(-21) + (-26)}{180} = \frac{-47}{180}$$

In performing additions of this type, you should set down your work as follows:

$$\frac{-7}{60} + \frac{-13}{90} = \frac{-7 \times 3}{60 \times 3} + \frac{-13 \times 2}{90 \times 2} = \frac{(-21) + (-26)}{180} = \frac{-47}{180}$$

Work out the L.C.D. on a separate sheet of paper.

Self-correcting Exercise #8

Answers may be found on page 58 of this lesson.

1. State the L.C.D. of each of the following pairs of fractions.

(a) $\frac{-5}{6}$ and $\frac{7}{18}$ __________ (b) $\frac{3}{14}$ and $\frac{8}{21}$ __________

(c) $\frac{-6}{25}$ and $\frac{-17}{30}$ __________ (d) $\frac{5}{36}$ and $\frac{11}{24}$ __________

(e) $\frac{4}{15}$ and $\frac{-9}{85}$ __________ (f) $\frac{3}{10}$ and $\frac{7}{30}$ __________

(g) $\frac{11}{12}$ and $\frac{-5}{18}$ __________ (h) $\frac{11}{28}$ and $\frac{3}{98}$ __________

2. Find the following sums. Answers should be expressed in lowest terms and as mixed numbers, where possible.

(a) $\frac{7}{12} + \frac{5}{8} = \frac{7 \times \quad}{12 \times \quad} + \frac{5 \times \quad}{8 \times \quad} = \frac{\quad + \quad}{\qquad} =$

(b) $\frac{-5}{16} + \frac{19}{24} = \frac{-5 \times}{16 \times} + \frac{19 \times}{24 \times} = \frac{\quad + \quad}{\qquad} =$

(c) $\frac{18}{25} - \frac{-21}{50} =$

(d) $\frac{3}{14} - \frac{5}{21} =$

F. <u>Complex Fractions</u>

A COMPLEX FRACTION is an expression whose numerator and denominator contain one or more fractions. For example, the three expressions given below are all complex fractions.

$$\frac{\frac{5}{6}}{\frac{7}{4}}, \quad \frac{\frac{1}{2}+\frac{1}{3}}{5}, \quad \frac{\frac{5}{9}\times\frac{-3}{7}}{\frac{-8}{15}\div\frac{-4}{5}}$$

When simplifying complex fractions, we can interpret the large horizontal line to be a sign of division. We first perform any operations that are indicated in the numerator or denominator of the complex fraction. In the last step, we perform the operation of division which is indicated by the long horizontal line. Every complex fraction can be simplified to the form $\frac{p}{q}$ where p and q are two integers ($q \neq 0$) which have no common factors other than 1 and -1.

Note how the following complex fractions can be simplified.

EXAMPLE 1:

$$\frac{\frac{5}{6}}{\frac{7}{4}} = \frac{5}{6} \div \frac{7}{4} = \frac{5}{6} \times \frac{4}{7} = \frac{5 \times \cancel{4}^{2}}{\cancel{6}_{3} \times 7} = \frac{\quad}{21}$$

THE LARGE HORIZONTAL LINE HAS BEEN REPLACED BY THE DIVISION SYMBOL "÷".

EXAMPLE 2:

$$\frac{\frac{1}{2}+\frac{1}{3}}{5} = \left(\frac{1}{2}+\frac{1}{3}\right) \div 5$$

DIVIDE BY 5.

Simplify within brackets first.

$$= \left(\frac{3+2}{6}\right) \div 5$$

MULTIPLY BY $\frac{1}{5}$.

$$= \frac{5}{6} \times \frac{1}{5} \qquad = \frac{\quad}{6}$$

Self-correcting Exercise #9

Answers may be found on page 59 of this lesson.

1. Write each complex fraction as a division question. (Do not complete the simplification.)

(a) $\dfrac{1}{\frac{1}{4}} = 1 \div \frac{1}{4}$

(b) $\dfrac{\frac{1}{3} - \frac{1}{5}}{7} =$

(c) $\dfrac{\frac{3}{4}}{\frac{2}{3}} =$

(d) $\dfrac{\frac{1}{5}}{3 \times \frac{2}{7}} =$

2. Simplify each complex fraction.

(a) $\dfrac{-5}{\frac{10}{21}} =$

(b) $\dfrac{\frac{3}{2} + \frac{8}{5}}{\frac{7}{5}} =$

EXERCISE - Operating With Rational Numbers

1. Fill in the blanks.

(a) The sum of $\frac{m}{c}$ and $\frac{n}{c}$ is $\frac{\_\_\_\_}{c}$.

(b) The difference of $\frac{m}{c}$ and $\frac{n}{c}$ is $\frac{\_\_\_\_}{c}$.

(c) The product of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{\_\_\_\_}{bd}$.

(d) The quotient of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{\_\_\_\_}{bc}$.

(e) The reciprocal of $\frac{2}{3}$ is _____.

(f) Dividing by $\frac{3}{4}$ is the same as __________ by __________.

(g) To add or subtract two rational numbers with unlike denominators, first transform each fraction into an equal fraction whose denominator is the __________ __________ denominator of the two fractions.

(h) The quotient of $\frac{3}{2}$ and $\frac{5}{7}$ is determined by multiplying $\frac{3}{2}$ by the ________________ of $\frac{5}{7}$.

(i) The L.C.D. of $\frac{7}{12}$ and $\frac{-3}{45}$ is ______________.

(j) To subtract two rational numbers with the same denominator, you subtract the two ____________________ and put this result over the common __________________.

(k) The product of two rational numbers can be found by taking the ______________ of the numerators and dividing this result by the product of the ________________________.

(l) In order to multiply the mixed numbers $-3\frac{1}{8}$ and $2\frac{5}{7}$, we must first change them to the rational numbers _____ and _____.

2. Perform the following operations. Express your answers in lowest terms and, where possible, as mixed numbers.

(a) $\frac{7}{8} + \frac{3}{8} = \frac{7+3}{8} = \frac{10}{8} = \frac{5}{4} = 1\frac{1}{4}$

(b) $\frac{-3}{16} + \frac{5}{16} =$

(c) $\frac{-7}{18} + \frac{-17}{18} =$

(d) $\frac{11}{15} + \frac{12}{15} =$

(e) $\frac{-11}{12} - \frac{15}{12} = \frac{-11 - 15}{12} =$

(f) $\frac{4}{3} + \frac{-8}{3} =$

(g) $\frac{17}{18} - \frac{-11}{18} = \frac{17 - (-11)}{18} =$

(h) $\frac{-3}{4} + \frac{3}{4} =$

(i) $\frac{-3}{8} \times \frac{-4}{9} = \frac{\overset{-1}{\cancel{-3}} \times \overset{-1}{\cancel{-4}}}{\underset{2}{\cancel{8}} \times \underset{3}{\cancel{9}}} = \frac{-1 \times -1}{2 \times 3} = \frac{1}{6}$

(j) $\frac{-11}{5} \times \frac{5}{11} =$

(k) $\frac{93}{5} \times \frac{-5}{31} =$

(l) $36 \times 1\frac{5}{6} =$

(m) $-2\frac{5}{9} \times 2\frac{1}{4} =$

(n) $2\frac{8}{9} \times 13\frac{1}{2} =$

(o) $\frac{-7}{5} \div \frac{21}{10} = \frac{-7}{5} \times \frac{10}{21} = \frac{-7 \times 10}{5 \times 21} =$

(p) $\frac{8}{9} \div 3 =$

(q) $-25 \div \frac{-10}{3} =$

(r) $-7\frac{1}{2} \div 6\frac{1}{4} =$

(s) $3\frac{5}{9} \div 3\frac{1}{5} =$

(t) $\frac{5}{18} - \frac{-1}{6} = \frac{5}{18} - \frac{-1 \times 3}{6 \times 3} = \frac{5}{18} - \frac{-3}{18} = \frac{5-(-3)}{18} = \frac{5+3}{18} = \_\_\_ = \_\_\_$

(u) $\frac{5}{36} + \frac{7}{30} =$

(v) $\frac{1}{60} - \frac{5}{120} =$

(w) $\frac{-5}{36} + \frac{7}{54} =$

(x) $\frac{-5}{6} + \frac{-5}{9} =$

3. Write each complex fraction in simplest terms.

(a) $\dfrac{-1}{\frac{1}{5}} = -1 \div \frac{1}{5} = -1 \times \frac{5}{1} =$

(b) $\dfrac{\frac{9}{4}}{3} =$

(c) $\dfrac{\frac{5}{8}}{\frac{15}{4}} =$

(d) $\dfrac{4 \times \frac{3}{2}}{\frac{1}{2}} =$

(e) $\dfrac{\frac{2}{3} + \frac{1}{4}}{4} = \left(\frac{2}{3} + \frac{1}{4}\right) \div 4 =$

(f) $\dfrac{\frac{1}{4}}{\frac{7}{8} - \frac{1}{8}} =$

Topic Three: Properties of the Rational Numbers

The set of rational numbers satisfies all the properties we attributed to the integers under the operations of addition and multiplication (see chart on page 45, lesson 3) besides having some new properties of its own.

A. Commutative Properties

Addition and multiplication are commutative in set Q, but subtraction and division are not.

B. Associative Properties

Addition and multiplication are associative in set Q, but subtraction and division are not.

C. Distributive Properties

Multiplication distributes over addition and subtraction in set Q.

D. Closure Properties

Like set I, set Q is closed under the operations of addition, subtraction, and multiplication. This means that when we add, subtract, or multiply two rational numbers, the result is always a rational number.

Unlike set I, set Q has the added feature that it is closed under the operation of division, except for division by zero. This means that when we divide two rational numbers (the divisor can't be zero), the result is always a rational number. Divide the following expressions and note that the results are always rational numbers.

1. $\frac{-3}{2} \div \frac{9}{4} =$

2. $6 \div \frac{12}{13} =$

3. $\frac{0}{5} \div \frac{8}{3} =$

E. Identity Elements

Like set I, set Q has identity elements for both addition and multiplication.

1. The additive identity in set Q is zero. Any rational number of the form $\frac{0}{x}$ (where $x \neq 0$) is equivalent to the additive identity. When zero is added to any rational number, that number is left unchanged.

   Fill in the blanks.

   $\frac{1}{2} + \frac{0}{2} =$ _____ $\qquad 0 + \frac{3}{4} =$ _____ $\qquad 0 +$ ____ $= \frac{-3}{8}$

2. The multiplicative identity in set Q is one. Any rational number of the form $\frac{x}{x}$ (where $x \neq 0$) is equivalent to the multiplicative identity. When any rational number is multiplied by one, that number is left unchanged.

   Fill in the blanks.

   $\frac{-6}{7} \times 1 =$ _____ $\qquad \frac{3}{4} \times \frac{2}{2} =$ _____ $\qquad \frac{-4}{3} \times$ ____ $= \frac{-4}{3}$

F. Inverse Elements

1. In our discussion of the set of integers in Lesson 3, we found that every integer has an additive inverse. Two integers that are additive inverses of each other have the same absolute value. The sum of an integer and its additive inverse is always zero.

   Fill in the blanks.

   3 and -3 are additive inverses since $3 + (-3) =$ _____.

   -5 and ____ are additive inverses since $-5 +$ ____ $= 0$.

   0 and ____ are additive inverses since $0 +$ ____ $=$ _____.

   Every rational number in set Q also has an additive inverse. For every rational number $\frac{a}{b}$, there is another rational number $\frac{-a}{b}$ which is called the additive inverse of $\frac{a}{b}$. Two rational numbers which are additive inverses of each other have the

same absolute value.

i.e. $\left|\frac{a}{b}\right| = \left|\frac{-a}{b}\right|$

The sum of a rational number and its additive inverse is always zero.

i.e. $\frac{a}{b} + \frac{-a}{b} = \frac{a + (-a)}{b} = \frac{0}{b} = 0$

Fill in the blanks.

$\frac{2}{3}$ and $\frac{-2}{3}$ are additive inverses since $\frac{2}{3} +$ ___ $= 0$

$\frac{-5}{7}$ and $\frac{5}{7}$ are additive inverses since $\frac{-5}{7} +$ ___ $=$ ___

$\frac{1}{4}$ and ___ are additive inverses since $\frac{1}{4} +$ ___ $=$ ___

2. In our discussion of the set of integers, no mention was made of multiplicative inverses. The rational numbers have a new property concerning multiplicative inverses.

For every rational number $\frac{a}{b}$, (except zero) there is another rational number $\frac{b}{a}$ which is called the MULTIPLICATIVE INVERSE of $\frac{a}{b}$. (You will recall that $\frac{b}{a}$ may also be called the RECIPROCAL of $\frac{a}{b}$.) Two rational numbers that are multiplicative inverses of each other have their numerators and denominators switched. The product of a rational number and its multiplicative inverse is always one.

i.e. $\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a} = 1$

Fill in the blanks.

$\frac{-2}{3}$ and $\frac{3}{-2}$ are multiplicative inverses since $\frac{-2}{3} \times$ ___ $= 1$

$\frac{1}{2}$ and 2 are multiplicative inverses since $\frac{1}{2} \times$ ___ $=$ ___

$\frac{4}{3}$ and ___ are multiplicative inverses since $\frac{4}{3} \times$ ___ $=$ ___

G. Summary-Properties of the Rationals Under Addition and Multiplication

For any three rational numbers $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$:

FILL IN THE BLANKS IN THE CHART.

	Addition	Multiplication
Closure Properties	$\left(\frac{a}{b} + \frac{c}{d}\right)$ is a rational number.	$\left(\frac{a}{b} \times \frac{c}{d}\right)$ is a ________ number.
Commutative Properties	$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \text{—}$	$\frac{a}{b} \times \frac{c}{d} = \text{—} \times \frac{a}{b}$
Associative Properties	$\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \text{—}$	$\frac{a}{b}\left(\frac{c}{d} \times \frac{e}{f}\right) = \left(\text{—} \times \frac{c}{d}\right)\frac{e}{f}$
Distributive Property	$\frac{a}{b}\left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} \times \text{—}\right) + \left(\text{—} \times \frac{e}{f}\right)$	
Identity Elements	Additive identity is zero. $\frac{a}{b} + 0 = \text{——}$	Multiplicative identity is one. $\frac{a}{b} \times 1 = \text{——}$
Inverse Elements	Additive inverse of $\frac{a}{b}$ is $\frac{-a}{b}$. $\frac{a}{b} + \text{—} = 0$	*Multiplicative inverse of $\frac{a}{b}$ is $\frac{b}{a}$. $\frac{a}{b} \times \text{—} = 1$

*This is a property of the rational numbers but not of the integers.

A set of numbers which has all of the above properties with respect to the two operations of addition and multiplication is called a FIELD.

The set of integers and the set of whole numbers lack some of the properties of a field. Multiplicative inverses do not exist in set I. The set of whole numbers, W, has neither additive inverses nor multiplicative inverses.

Thus, the set of rational numbers forms a field with respect to the operations of addition and multiplication while the other sets of numbers we have studied up to this point do not.

EXERCISE - Properties of Set Q

1. Fill in the blanks.

   (a) The ____________________ inverse of $\frac{3}{5}$ is $\frac{-3}{5}$.

   (b) The result of multiplying any rational number by zero is ________________.

   (c) Multiplication distributes over ______________ and ______________ in set Q.

   (d) When the product of two rational numbers is one, each is the ______________ ______________ of the other.

   (e) The operations of ______________ and ______________ are not commutative or associative in set Q.

   (f) $\frac{-3}{-3}$ is a form of the ______________ identity.

   (g) When the sum of two rational numbers is ______________, each is the additive inverse of the other.

   (h) Set Q is closed under the operation of division except for division by ______________.

   (i) $\frac{0}{-9}$ is a form of the ______________ identity.

2. State the additive inverse and the multiplicative inverse of each of the following rational numbers.

	Additive Inverse	Multiplicative Inverse
(a) $\frac{-2}{5}$	$\frac{2}{5}$	$\frac{5}{-2}$
(b) $\frac{6}{11}$		

	Additive Inverse	Multiplicative Inverse
(c) $-3\frac{1}{2}$ (or $\frac{-7}{2}$)	$3\frac{1}{2}$	$\frac{-2}{7}$
(d) $4\frac{1}{4}$		
(e) 3		
(f) $\frac{-1}{7}$		
(g) $\frac{y}{x + z}$		

3. Decide if each of the following statements is true or false. Give a reason for your answer. (Review exercise #2 on page 30, lesson 2.)

(a) $\left(\frac{1}{2} \times \frac{2}{3}\right)$ is a rational number. true

Set Q is closed under multiplicatcon.

(b) $\frac{2}{3} \div \frac{-1}{2} = \frac{-1}{2} \div \frac{2}{3}$ ______

Division is not commutative.

(c) $\frac{3}{4} + 0 = \frac{3}{4}$ ______

____

(d) $\frac{3}{8} - \frac{1}{4} = \frac{1}{4} - \frac{3}{8}$ ______

____

(e) $\left(\frac{1}{2} + \frac{-1}{4}\right) + \frac{-1}{3} = \frac{1}{2} + \left(\frac{-1}{4} + \frac{-1}{3}\right)$ ________

____

(f) $\frac{3}{4} + \frac{-3}{4} = 0$ ________

____

(g) $\left(\frac{-5}{4} + \frac{1}{2}\right)$ is a rational number. ________

____

(h) $\left(\frac{3}{4} \div \frac{1}{5}\right) \div \frac{1}{7} = \frac{3}{4} \div \left(\frac{1}{5} \div \frac{1}{7}\right)$ ________

____

(i) $1 \times \frac{6}{7} = \frac{6}{7}$ ________

____

(j) $\frac{1}{2}\left(6 - 3\right) = \left(\frac{1}{2} \times 6\right) - \left(\frac{1}{2} \times 3\right)$ ________

____

(k) $\frac{5}{2} \times \frac{2}{5} = 1$ ________

____

(l) $\frac{-2}{3} + \frac{1}{4} = \frac{1}{4} + \frac{-2}{3}$ ________

____

Key to Self-correcting Exercises in Lesson 4

Exercise #1, page 6

1. (a) $(-5) \div (-3) = \frac{-5}{-3} = \frac{5}{3}$

PLACE MINUS SIGN IN NUMERATOR

(b) $2 \div (-3) = \frac{2}{-3} = \frac{-2}{3}$

(c) $(-8) \div (-9) = \frac{-8}{-9} = \frac{8}{9}$

(d) $3 \div (-5) = \frac{3}{-5} = \frac{-3}{5}$

2. (a) $\frac{-3}{4}$ is negative since one term is negative.

(b) $\frac{0}{5}$ is zero since the numerator is zero.

(c) $\frac{-3}{-8}$ is positive since both terms are negative.

(d) $\frac{4}{5}$ is positive since both terms are positive.

(e) $\frac{6}{0}$ is undefined since the denominator is zero.

(f) $\frac{2}{-3}$ is negative since one term is negative.

Exercise #2, page 12

1. (a) $15 = 3 \times 5$

(b) $30 = 5 \times 6$
$= 5 \times 2 \times 3$

(c) $32 = 4 \times 8$
$= 2 \times 2 \times 2 \times 2 \times 2$

(d) $63 = 7 \times 9$
$= 7 \times 3 \times 3$

(e) $54 = 6 \times 9$
$= 2 \times 3 \times 3 \times 3$

(f) $105 = 5 \times 21$
$= 5 \times 3 \times 7$

Note: In the questions above, your original choice of factors may differ from those given. This will result in the prime factors appearing in a different order in the final answers. Just check to see that the prime factors that you have listed are identical to those given here.

2.

	Number Pair	Product of Primes	H. C. F.
(a)	48, 168	$48 = 2 \times 2 \times 2 \times 2 \times 3$ $168 = 2 \times 2 \times 2 \times 3 \times 7$	$2 \times 2 \times 2 \times 3$ $= \underline{24}$
(b)	30, 42	$30 = 2 \times 3 \times 5$ $42 = 2 \times 3 \times 7$	2×3 $= \underline{6}$
(c)	24, 96	$24 = 2 \times 2 \times 2 \times 3$ $96 = 2 \times 2 \times 2 \times 2 \times 2 \times 3$	$2 \times 2 \times 2 \times 3$ $= \underline{24}$
(d)	66, 165	$66 = 2 \times 3 \times 11$ $165 = 3 \times 5 \times 11$	3×11 $= \underline{33}$

3. (a) $\frac{-48}{168} = \frac{-48 \div 24}{168 \div 24} = \frac{-2}{7}$

H.C.F. IS 6.

(b) $\frac{30}{42} = \frac{30 \div 6}{42 \div 6} = \frac{5}{7}$

(c) $\frac{24}{96} = \frac{24 \div 24}{96 \div 24} = \frac{1}{4}$

(d) $\frac{-66}{165} = \frac{-66 \div 33}{165 \div 33} = \frac{-2}{5}$

4. (a) $75 = 3 \times 5 \times 5$
$225 = 3 \times 3 \times 5 \times 5$

H.C.F. is 75.

$\frac{-75}{225} = \frac{-75 \div 75}{225 \div 75} = \frac{-1}{3}$

(b) $70 = 2 \times 5 \times 7$
$105 = 3 \times 5 \times 7$

H.C.F. is 35.

$\frac{70}{105} = \frac{70 \div 35}{105 \div 35} = \frac{2}{3}$

(c) $60 = 2 \times 2 \times 3 \times 5$
$108 = 2 \times 2 \times 3 \times 3 \times 3$

H.C.F. is 12.

$\frac{60}{108} = \frac{60 \div 12}{108 \div 12} = \frac{5}{9}$

(d) $40 = 2 \times 2 \times 2 \times 5$
$80 = 2 \times 2 \times 2 \times 2 \times 5$

H.C.F. is 40.

$\frac{-40}{80} = \frac{-40 \div 40}{80 \div 40} = \frac{-1}{2}$

Exercise #3, page 17

1.

	$\frac{a}{b}$, $\frac{c}{d}$	Cross Products		Comparison of Cross Products	Ordering of the Fractions
		ad	bc		
(a)	$\frac{-4}{3}$, $\frac{-5}{2}$	-4×2 $= \underline{-8}$	3×-5 $= \underline{-15}$	(BOTH SYMBOLS $-8 > -15$	POINT TO THE RIGHT) $\frac{-4}{3} > \frac{-5}{2}$
(b)	$\frac{3}{14}$, $\frac{5}{26}$	3×26 $= \underline{78}$	14×5 $= \underline{70}$	$78 > 70$	$\frac{3}{14} > \frac{5}{26}$
(c)	$\frac{2}{3}$, $\frac{-2}{3}$	2×3 $= 6$	3×-2 $= -6$	$6 > -6$ (since 6 lies to the right of -6)	$\frac{2}{3} > \frac{-2}{3}$
(d)	$\frac{-5}{6}$, $\frac{-3}{4}$	-5×4 $= -20$	6×-3 $= -18$	$-20 < -18$ (since -20 lies to the left of -18.)	$\frac{-5}{6} < \frac{-3}{4}$
(e)	$\frac{7}{19}$, $\frac{2}{5}$	7×5 $= 35$	19×2 $= 38$	$35 < 38$	$\frac{7}{19} < \frac{2}{5}$

2. (a) $\frac{4}{5} < \frac{5}{6}$

since $(4 \times 6) < (5 \times 5)$.

(b) $\frac{-3}{2} < \frac{11}{8}$

since $(-3 \times 8) < (2 \times 11)$.

(c) $\frac{2}{3} = \frac{10}{15}$

since $(2 \times 15) = (3 \times 10)$.

(d) $\frac{3}{6} > \frac{-1}{2}$

since $(3 \times 2) > (6 \times -1)$.

(e) $\frac{10}{4} = \frac{5}{2}$

since $(10 \times 2) = (4 \times 5)$.

(f) $\frac{-11}{12} < \frac{-5}{6}$

since $(-11 \times 6) < (12 \times -5)$.

Exercise #4, page 21

1. (a) $\frac{13}{2} = 6\frac{1}{2}$

(13 ÷ 2 = 6, remainder 1)

(b) $\frac{-35}{3} = -11\frac{2}{3}$

(35 ÷ 3 = 11, remainder 2)

(c) $\frac{17}{6} = 2\frac{5}{6}$

(17 ÷ 6 = 2, remainder 5)

(d) $\frac{-60}{7} = -8\frac{4}{7}$

(60 ÷ 7 = 8, remainder 4.)

2. (a) $5\frac{3}{8} = \frac{43}{8} \leftarrow (5\times 8)+3$

(b) $-1\frac{5}{7} = \frac{-12}{7} \leftarrow -(1\times 7)+5$

(c) $-3\frac{2}{3} = \frac{-11}{3} \leftarrow (3\times 3)+2$

(d) $6\frac{2}{5} = \frac{32}{5} \leftarrow (6\times 5)+2$

3. (a) $\frac{45}{36} = \frac{45 \div 9}{36 \div 9} = \frac{5}{4} = 1\frac{1}{4}$.

(b) $\frac{-18}{4} = \frac{-18 \div 2}{4 \div 2} = \frac{-9}{2} = -4\frac{1}{2}$

(c) $\frac{-96}{56} = \frac{-96 \div 8}{56 \div 8} = \frac{-12}{7} = -1\frac{5}{7}$

(d) $\frac{80}{15} = \frac{80 \div 5}{15 \div 5} = \frac{16}{3} = 5\frac{1}{3}$

Exercise #5, page 29

1. (a) $\frac{2}{3} \times \frac{15}{16} = \frac{\overset{1}{\cancel{2}} \times \overset{5}{\cancel{15}}}{\underset{1}{\cancel{3}} \times \underset{8}{\cancel{16}}}$

(b) $\frac{3}{8} \times \frac{4}{9} = \frac{\overset{1}{\cancel{3}} \times \overset{1}{\cancel{4}}}{\underset{2}{\cancel{8}} \times \underset{3}{\cancel{9}}}$

(c) $\frac{3}{14} \times \frac{7}{8} = \frac{3 \times \overset{1}{\cancel{7}}}{\underset{2}{\cancel{14}} \times 8}$

(d) $\frac{-7}{8} \times \frac{4}{21} = \frac{\overset{-1}{\cancel{-7}} \times \overset{1}{\cancel{4}}}{\underset{2}{\cancel{8}} \times \underset{3}{\cancel{21}}}$

(e) $\frac{-4}{15} \times \frac{-5}{8} = \frac{\overset{-1}{\cancel{-4}} \times \overset{-1}{\cancel{-5}}}{\underset{3}{\cancel{15}} \times \underset{2}{\cancel{8}}}$

(f) $\frac{21}{25} \times \frac{-5}{7} = \frac{\overset{3}{\cancel{21}} \times \overset{-1}{\cancel{-5}}}{\underset{5}{\cancel{25}} \times \underset{1}{\cancel{7}}}$

2. (a) $\frac{8}{3} \times \frac{3}{5} = \frac{8 \times \overset{1}{\cancel{3}}}{\underset{1}{\cancel{3}} \times 5} = \frac{8 \times 1}{1 \times 5} = \frac{8}{5} = 1\frac{3}{5}$

Exercise #5, (cont'd)

2. (b) $\frac{-15}{4} \times \frac{16}{3} = \frac{\overset{-5}{\cancel{-15}} \times \overset{4}{\cancel{16}}}{\underset{1}{\cancel{4}} \times \underset{1}{\cancel{3}}} = \frac{-5 \times 4}{1 \times 1} = -20$

(c) $\frac{52}{17} \times \frac{34}{39} = \frac{\overset{4}{\cancel{52}} \times \overset{2}{\cancel{34}}}{\underset{1}{\cancel{17}} \times \underset{3}{\cancel{39}}} = \frac{4 \times 2}{1 \times 3} = \frac{8}{3} = 2\frac{2}{3}$

3. (a) $-6\frac{1}{4} \times 2\frac{4}{5} = \frac{-25}{4} \times \frac{14}{5} = \frac{\overset{-5}{\cancel{-25}} \times \overset{7}{\cancel{14}}}{\underset{2}{\cancel{4}} \times \underset{1}{\cancel{5}}} = \frac{-5 \times 7}{2 \times 1} = \frac{-35}{2} = -17\frac{1}{2}$

(b) $2\frac{5}{9} \times 2\frac{1}{4} = \frac{23}{9} \times \frac{9}{4} = \frac{23 \times \overset{1}{\cancel{9}}}{\underset{1}{\cancel{9}} \times 4} = \frac{23 \times 1}{1 \times 4} = \frac{23}{4} = 5\frac{3}{4}$

Exercise #6, page 32

1. (a) $\frac{3}{8}$ (b) $\frac{1}{2}$ (c) $\frac{12}{-5}$

(d) $\frac{1}{-3}$ (e) $\frac{-5}{2}$ (f) $\frac{4}{1}$ (or 4)

GIVE RECIPROCAL OF DIVISOR

2. (a) $\frac{2}{3} \div \frac{1}{4} = \frac{2}{3} \times \frac{4}{1}$

(b) $5 \div \frac{3}{7} = \frac{5}{1} \times \frac{7}{3}$

(c) $\frac{2}{3} \div 8 = \frac{2}{3} \times \frac{1}{8}$

(d) $\frac{6}{5} \div \frac{-3}{2} = \frac{6}{5} \times \frac{2}{-3}$

(e) $\frac{1}{8} \div \frac{1}{6} = \frac{1}{8} \times \frac{6}{1}$

(f) $\frac{-1}{4} \div -5 = \frac{-1}{4} \times \frac{1}{-5}$

GIVE RECIPROCAL OF DIVISOR

3. (a) $\frac{2}{3} \div \frac{-1}{6} = \frac{2}{3} \times \frac{6}{-1} = \frac{2 \times \overset{2}{\cancel{6}}}{\underset{1}{\cancel{3}} \times -1} = \frac{2 \times 2}{1 \times -1} = \frac{4}{-1} = -4$

(b) $\frac{11}{12} \div \frac{3}{4} = \frac{11}{12} \times \frac{4}{3} = \frac{11 \times \overset{1}{\cancel{4}}}{\underset{3}{\cancel{12}} \times 3} = \frac{11 \times 1}{3 \times 3} = \frac{11}{9} = 1\frac{2}{9}$

(c) $\frac{7}{4} \div \frac{3}{16} = \frac{7}{4} \times \frac{16}{3} = \frac{\cancel{7} \times \cancel{16}^{4}}{\cancel{4}_{1} \times 3} = \frac{7 \times 4}{1 \times 3} = \frac{28}{3} = 9\frac{1}{3}$

(d) $\frac{-2}{3} \div 8 = \frac{-2}{3} \times \frac{1}{8} = \frac{\cancel{-2}^{-1} \times 1}{3 \times \cancel{8}_{4}} = \frac{-1 \times 1}{3 \times 4} = \frac{-1}{12}$

GIVE RECIPROCAL OF DIVISOR.

4. (a) $3\frac{3}{5} \div \frac{9}{10} = \frac{18}{5} \div \frac{9}{10} = \frac{18}{5} \times \frac{10}{9} = \frac{\cancel{18}^{2} \times \cancel{10}^{2}}{\cancel{5}_{1} \times \cancel{9}_{1}} = \frac{2 \times 2}{1 \times 1} = 4$

(b) $-7 \div 4\frac{1}{5} = -7 \div \frac{21}{5} = \frac{-7}{1} \times \frac{5}{21} = \frac{\cancel{-7}^{-1} \times 5}{1 \times \cancel{21}_{3}} = \frac{-1 \times 5}{1 \times 3} = \frac{-5}{3} = -1\frac{2}{3}$

(c) $4\frac{1}{8} \div 11 = \frac{33}{8} \times \frac{1}{11} = \frac{\cancel{33}^{3} \times 1}{8 \times \cancel{11}_{1}} = \frac{3 \times 1}{8 \times 1} = \frac{3}{8}$

Exercise #7, page 37

1.

	Fraction Pairs	Denominators Expressed as Product of Primes	L.C.D.
(a)	$\frac{3}{108}$, $\frac{5}{72}$	$108 = 2 \times 2 \times 3 \times 3 \times 3$ $72 = 2 \times 2 \times 2 \times 3 \times 3$	GREATEST NO. OF TIMES 3 OCCURS IN ONE NUMBER $(3 \times 3 \times 3) \times (2 \times 2 \times 2)$ $= 216$ GREATEST NO. OF TIMES 2 OCCURS IN ONE NUMBER
(b)	$\frac{5}{6}$, $\frac{-8}{9}$	$6 = 2 \times 3$ $9 = 3 \times 3$	$2 \times 3 \times 3$ $= 18$
(c)	$\frac{7}{24}$, $\frac{5}{18}$	$24 = 2 \times 2 \times 2 \times 3$ $18 = 2 \times 3 \times 3$	$2 \times 2 \times 2 \times 3 \times 3$ $= 72$
(d)	$\frac{7}{15}$, $\frac{10}{27}$	$15 = 3 \times 5$ $27 = 3 \times 3 \times 3$	$3 \times 3 \times 3 \times 5$ $= 135$

2.

Fraction Pairs	Step 1	Step 2	Step 3
$\frac{3}{16}$ and $\frac{-5}{36}$	$16 = 2\times 2\times 2\times 2$ $36 = 2\times 2\times 3\times 3$	$2\times 2\times 2\times 2\times 3\times 3$ $= 144$	$\frac{3}{16} = \frac{3\times 9}{16\times 9} = \frac{27}{144}$ $\frac{-5}{36} = \frac{-5\times 4}{36\times 4} = \frac{-20}{144}$
$\frac{-7}{60}$ and $\frac{-13}{90}$	$60 = 2\times 2\times 3\times 5$ $90 = 2\times 3\times 3\times 5$	$2\times 2\times 3\times 3\times 5$ $= 180$	$\frac{-7}{60} = \frac{-7\times 3}{60\times 3} = \frac{-21}{180}$ $\frac{-13}{90} = \frac{-13\times 2}{90\times 2} = \frac{-26}{180}$
$\frac{3}{56}$ and $\frac{-20}{98}$	$56 = 2\times 2\times 2\times 7$ $98 = 2\times 7\times 7$	$2\times 2\times 2\times 7\times 7$ $= 392$	$\frac{3}{56} = \frac{3\times 7}{56\times 7} = \frac{21}{392}$ $\frac{-20}{98} = \frac{-20\times 4}{98\times 4} = \frac{-80}{392}$
$\frac{-5}{18}$ and $\frac{7}{24}$	$18 = 2\times 3\times 3$ $24 = 2\times 2\times 2\times 3$	$2\times 2\times 2\times 3\times 3$ $= 72$	$\frac{-5}{18} = \frac{-5\times 4}{18\times 4} = \frac{-20}{72}$ $\frac{7}{24} = \frac{7\times 3}{24\times 3} = \frac{21}{72}$

Exercise #8, page 39

1. (a) 18 (b) 42 (c) 150 (d) 72

(e) 255 (f) 30 (g) 36 (h) 196

2. (a) $\frac{7}{12} + \frac{5}{8} = \frac{7\times 2}{12\times 2} + \frac{5\times 3}{8\times 3} + \frac{14+15}{24} = \frac{29}{24} = 1\frac{5}{24}$

(b) $\frac{-5}{16} + \frac{19}{24} = \frac{-5\times 3}{16\times 3} + \frac{19\times 2}{24\times 2} = \frac{(-15+38)}{48} = \frac{23}{48}$

(c) $\frac{18}{25} - \frac{-21}{50} = \frac{18 \times 2}{25 \times 2} - \frac{-21}{50} = \frac{36 - (-21)}{50} = \frac{36 + 21}{50} = \frac{57}{50} = 1\frac{7}{50}$

(d) $\frac{3}{14} - \frac{5}{21} = \frac{3 \times 3}{14 \times 3} - \frac{5 \times 2}{21 \times 2} = \frac{9 - 10}{42} = \frac{-1}{42}$

<u>Exercise #9, page 41</u>

1. (a) $\dfrac{1}{\frac{1}{4}} = 1 \div \frac{1}{4}$

THIS LINE MEANS "DIVIDE"

(b) $\dfrac{\frac{1}{3} - \frac{1}{5}}{7} = \left(\frac{1}{3} - \frac{1}{5}\right) \div 7$

(c) $\dfrac{\frac{3}{4}}{\frac{2}{3}} = \frac{3}{4} \div \frac{2}{3}$

(d) $\dfrac{\frac{1}{5}}{3 \times \frac{2}{7}} = \frac{1}{5} \div \left(3 \times \frac{2}{7}\right)$

2. (a) $\dfrac{-5}{\frac{10}{21}} = -5 \div \frac{10}{21}$

$= \frac{-5}{1} \times \frac{21}{10}$

$= \frac{\overset{-1}{\cancel{-5}} \times 21}{1 \times \underset{2}{\cancel{10}}}$

$= \frac{-21}{2}$

$= -10\frac{1}{2}$

(b) $\dfrac{\frac{3}{2} + \frac{8}{5}}{\frac{7}{5}} = \left(\frac{3}{2} + \frac{8}{5}\right) \div \frac{7}{5}$

$= \left(\frac{15 + 16}{10}\right) \div \frac{7}{5}$

$= \frac{31}{10} \times \frac{5}{7}$

$= \frac{31 \times \overset{1}{\cancel{5}}}{\underset{2}{\cancel{10}} \times 7}$

$= \frac{31}{14}$

$= 2\frac{3}{14}$

End of Lesson 4

Lesson 5

Decimal Numbers

Basic Algebra and Geometry

DECIMAL NUMBERS

Topic One: Meaning of Decimal Numbers

A. Base-ten Numeration

A system for writing numerals is called a NUMERATION SYSTEM. The numeration system that we use is called the Hindu-Arabic system. In this system, there are ten symbols to work with and the grouping is by tens. Hence, the Hindu-Arabic system is often called BASE-TEN NUMERATION or DECIMAL NUMERATION. (The word "decimal" comes from the Latin word for "ten".)

You are familiar with the ten symbols or digits used in base-ten numeration. These are:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

For writing numbers greater than 9, more than one of these digits must be used. The way in which digits are combined to name numbers greater than 9 involves the property of place value. According to this property, the number represented by each digit in a numeral is determined by the place that digit occupies in the numeral.

The digits in a base-ten numeral are grouped in threes, going from right to left. Spaces are used to separate the groups. Each of the groupings is called a PERIOD and each period has a special name. Only the period farthest to the left in a numeral can contain less than three digits.

Use spaces to separate the following base-ten numerals into periods. Work from right to left. (The first one has been done for you.)

8643970042	123478900462
8 643 970 042	______
46322875	8933607
______	______
6637248983 2	235562831
______	______

Each of the periods in a numeral has a special name. Study the names of the periods given in the chart below.

	billions	millions	thousands	units
1.	12	506	302	500
2.	150	200	000	097
3.		35	072	903

When we read a base-ten numeral, we give the digits in the period farthest to the left and then give the name of the period. We do the same for the other periods until we reach the period farthest to the right. No name is attached to this period.

1. The first numeral in the chart above is 12 506 302 500. What digits are in the billions period? 12 In the thousands period? _____ In the millions period? _____ This number is read, "12 billion, 506 million, 302 thousand, 500."

2. The second numeral in the chart is ______________________. What digits are in the billions period? ________ In the units period? 097 In the thousands period? _______. This number is read, "150 ____________, 200 ____________, 97."

3. The third numeral in the chart is ______________________. What digits are in the billions period? _____ In the thousands period? _____ In the millions period? _____. This number is read, "35 ____________, 72 ____________, 903."

Special values are assigned to each of the individual places in a numeral. Study the place values given in the chart on page 3. Notice that each place has a value ten times as great as the next place on the right. The value of any particular digit in a numeral can be found by multiplying the digit by the value of the place it holds.

	BILLIONS			MILLIONS			THOUSANDS			UNITS		
	hundred billions (100,000,000,000.)	ten billions (10,000,000,000.)	billions (1,000,000,000)	hundred millions (100,000,000)	ten millions (10,000,000.)	millions (1,000,000.)	hundred thousands (100,000)	ten thousands (10,000)	thousands (1000)	hundreds (100)	tens (10)	ones (1)
1.		3	1	0	6	5	2	7	9	0	4	8
2.					2	3	0	0	9	6	7	4
3.							5	6	0	0	0	7

1. The first number in the chart is 31 065 279 048.

   (i) The digit 3 is in the ten-billions place. It represents the number $(3 \times 10\ 000\ 000\ 000)$ or 30 000 000 000.

   (ii) The digit 1 is in the ________ place. It represents the number $(1 \times 1\ 000\ 000\ 000)$ or ________.

   (iii) The digit 6 is in the ________ place. It represents the number (__ $\times$ 10 000 000) or ________.

2. The second number in the chart is ________.

   (i) The digit __ is in the millions place. It represents the number (__ $\times$ 1 000 000) or ________.

   (ii) The digit __ is in the hundred-thousands place. It represents the number (__ $\times$ 100 000) or 0.

   (iii) The digit ___ is in the thousands place. It represents the number (__ $\times$ ________) or ________.

3. The third number in the chart is ________.

   (i) The digit 5 is in the ________ place. It represents the number (5 $\times$ ________) or ________.

   (ii) The digit ___ is in the thousands place. It represents the number (__ $\times$ 1 000) or ________.

   (iii) The digit 7 is in the ________ place. It represents the number (7 $\times$ ___) or ________.

Sometimes you must translate a number idea from words into digits. Use the following procedure:

1. Pick out the names of the various periods and note the digits that must appear in these periods.

2. Use a zero if no other digit is given to hold a particular place in a number.

3. Leave a space after the digits in each period.

EXAMPLE: Use digits to write the number "thirty-five billion, two hundred forty-seven million, twenty-three thousand, eight".
Solution

The digits 35 must appear in the billions period.

$\underline{35}$ ■■■■■■■■■

The digits 247 must appear in the millions period.

35 $\underline{247}$ ■■■■■■

The digits 23 must appear in the last two places of the thousands period. A zero must be used to hold the place of the hundred thousands.

35 247 $\underline{023}$ ■■■

The digit 8 must appear in the last place of the units period. Zeros must be used to hold the place of the hundreds and the tens.

35 247 023 $\underline{008}$

Thus, the numeral is 35 247 023 008.

Self-correcting Exercise #1

Answers may be found on page 48 of this lesson.

1. Fill in the blanks.

(a) Consider the number "sixty-eight million, thirty-two."

(i) The digits 68 are in the ________________ period.

(ii) The digits 32 are in the last two places of the ____________ period.

(iii) The number written in digits is ________________. Zeros have been used as placeholders in the entire ____________ period and in the ________________ place.

(b) Consider the number "five billion, fifty-six million, six thousand, one hundred twenty-five."

(i) The digit ___ is in the billions period.

(ii) The digits 56 are in the last two places of the ____________________ period.

(iii) The digit 6 is in the last place of the ____________________ period.

(iv) The digits _______ are in the units period.

(v) The number written in digits is ______________________. Zeros have been used as placeholders in the ____________ ____________ place, the ______________________ place, and the ______________________ place.

2. Use digits to write each of the following numbers.

(a) eighty-three million, six hundred nine thousand, fifty.

____

(b) sixty-billion, four million, two hundred thousand, sixty-nine.

____

(c) two million, eight thousand

____

(d) nine hundred million, forty-five thousand, three

____

(e) six billion, eight hundred thirty-seven thousand

____

B. Expanded Form

A numeral such as 63 209 is a very compact way to name a number. This compact notation, sometimes called standard notation, while very convenient for practical purposes, conceals the principles involved.

Any base ten number can also be expressed as a sum. Each term of the sum can be found by multiplying each individual digit in the number by the place that digit holds. When a numeral is written as a sum, it is said to be in EXPANDED FORM. In the study of numeration, expanded form is important because it exhibits the meaning of each digit in a numeral.

EXAMPLE: Write 63 209 in expanded form.

Solution

In the numeral 63 209:

- the digit 6 represents 6 ten-thousands or $(6 \times 10\ 000)$.
- the digit 3 represents 3 __________ or $(3 \times \_\_\_\_)$.
- the digit 2 represents 2 __________ or (2×100).
- the digit 0 represents 0 __________ or $(0 \times \_\_)$.
- the digit 9 represents 9 ones or $(9 \times \_\_)$.

This numeral could be expressed in expanded form by writing the sum of these place values.

i.e. $63\ 209 = (6 \times 10\ 000) + (3 \times 1\ 000) + (2 \times 100) + (0 \times 10) + (9 \times 1)$

Self-correcting Exercise #2

Answers may be found on page 48 of this lesson.

1. Write each numeral in expanded form.

   (a) 1 560 = (__ × _____) + (__ × _____) + (__ × _____) + (__ × _____)

   (b) 164 397 = (__ × _____) + (__ × _____) + (__ × _____) + (__ × _____)
   + (__ × _____) + (__ × _____)

   (c) 3 067 259 = (___ × __________) + (__ × __________) + (__ × _________)
   + (__ × _____) + (__ × _____) + (__ × _____) + (__ × _____)

2. Represent each expanded form by a numeral.

   (a) (4 × 100 000) + (3 × 10 000) + (0 × 1 000) + (5 × 100)
   + (2 × 10) + (0 × 1) = ____________________

   (b) (8 × 1 000 000) + (0 × 100 000) + (5 × 10 000) + (0 × 1 000)
   + (3 × 100) + (3 × 10) + (3 × 1) = ____________________

   (c) (5 × 10 000 000) + (5 × 1 000 000) + (1 × 100 000) + (0 × 10 000)
   + (0 × 1 000) + (2 × 100) + (0 × 10) + (5 × 1) = ____________________

C. Meaning of Decimals

So far in this lesson, we have restricted our discussion of base-ten numerals to natural numbers. To extend the idea of base-ten numeration so that numerals for all rational numbers can be given, we must identify places to the right of the ones place. In doing so, we continue to use the principles already established for base-ten numeration:

1. Each place has a value ten times as great as the next place on the right.

2. The value of any particular digit in a numeral can be found by multiplying the digit by the value of the place it holds.

A dot (or decimal point) is always placed between the digit in the ones place and the next digit on the right. The decimal point serves as a point of reference in establishing the place values of the various digits in a decimal number. DECIMAL POINT GOES HERE.

	ten thousands (10,000)	thousands (1000)	hundreds (100)	tens (10)	ones (1)	tenths ($\frac{1}{10}$)	hundredths ($\frac{1}{100}$)	thousandths ($\frac{1}{1000}$)	ten-thousandths ($\frac{1}{10,000}$)
1.	1	0	3	4	5 .	6	7		
2.				6	2 .	0	0	8	
3.		3	0	4	1 .	2	5	7	9

Note how the names of the places to the right and left of the ones place correspond. The only difference occurs in the endings of the names. The names of all the places to the left of the ones end in "s" while the names of all the places to the right of the ones end in "ths".

EXAMPLES:

(i) The first place to the left of the ones is the tens place, while the first place to the right of the ones is the tenths place.

(ii) The second place to the left of the ones is the hundreds place while the second place the right of the ones is the ____________ place.

(iii) The third place to the left of the ones is the ____________ place, while the third place to the right of the ones is the ____________ place.

1. The first number in the chart above is 10 345.67.

   (i) The digit 1 is in the ten-thousands place. It represents the number ($1 \times 10\ 000$) or 10 000.

   (ii) The digit 0 is in the ______________ place. It represents the number ($0 \times 1\ 000$) or 0.

   (iii) The digit 3 is in the ______________ place. It represents the number (__ $\times$ 100) or ____________.

(iv) The digit 4 is in the ______________ place. It represents the number (___ × ___) or __________.

(v) The digit 5 is in the ______________ place. It represents the number (___ × ___) or __________.

(vi) The digit 6 is in the ______________ place. It represents the number $\left(6 \times \frac{1}{10}\right)$ or $\frac{6}{10}$.

(vii) The digit 7 is in the ______________ place. It represents the number $\left(7 \times \frac{1}{100}\right)$ or __________.

2. The second number in the chart on page 8 is ________________.

(i) The digit ___ is in the tens place. It represents the number (___ × 10) or __________.

(ii) The digit ___ is in the ones place. It represents the number (___ × ___) or __________.

(iii) The digit ___ is in the thousandths place. It represents the number $\left(\_\_\_ \times \frac{1}{1000}\right)$ or $\frac{\_\_\_}{1000}$

(iv) Zeros appear in the tenths place and the ______________ place.

(v) The decimal point appears between the ______________ place and the ______________ place.

3. The third number in the chart is ________________.

(i) What digit is in the hundreds place? _____ In the thousandths place? _____ In the ten-thousandths place? _____ In the thousands place? _____ In the tenths place? _____ In the tens place? _____

(ii) The digit 5 is in the __________________ place. It represents the number (5 × ____) or ___________.

(iii) The decimal point appears between the digit _____ in the ones place and the digit _____ in the tenths place.

Self-correcting Exercise #3

Answers may be found on page 48 of this lesson.

1. In a decimal number, each place has a value ten times as great as the next place to the right and one-tenth as much as the next place to the left. Use this information to fill in the blanks below.

   (a) 1 thousand = _____ hundreds

   (b) 1 tenth = _____ hundredths

   (c) 1 thousandth = _____ ten-thousandths

   (d) 1 hundred = $\frac{1}{10}$ of 1 _______________

   (e) 1 ten = _____ of 1 hundred

   (f) 1 thousandth = $\frac{1}{10}$ of 1 ___________

   (g) _____ ones = 1 ten

   (h) 1 one = _____ tenths

2. Fill in the blanks.

   (a) In 32.915, the digit 9 is in the ______________________ place and represents $\frac{9}{10}$.

   (b) In 5.764 2, the digit 4 is in the ______________________ place and represents ________.

   (c) In 8.653 2, the digit 5 is in the ______________________ place and represents ________.

   (d) In 25.6, the digit 6 is in the ______________________ place and represents ________.

   (e) In 12.643 7, the digit 7 is in the ______________________ place and represents ________.

D. <u>Reading and Writing Decimal Numbers</u>

Use the following procedure when reading decimal numbers:

1. Read the whole number part (as you were shown on page 2 of this lesson).
2. Say "decimal" when you get to the decimal point.
3. Name the digits in order in the decimal part of the number.

EXAMPLES

25.13	is read "twenty-five, decimal one, three".
120.006	is read "one hundred twenty, decimal zero, zero, six".
7.165	is read "seven, _______ one, six, ______.
60.04	is read "______, decimal _______, four.
235.92	is read "two _______ thirty-five, decimal _______, _______."

Often you must translate a decimal number from words into symbols, when doing this, use the following procedure:

1. Write the whole number part as you were shown on page 4 of this lesson
2. Insert a decimal point in the number where the word "decimal" occurs in the expression.
3. Write the digits in the decimal part of the number, and insert a space after every third digit. (If there are only 4 digits after the decimal point, you may or may not leave a space between the thousandths and ten thousandths place.)

"sixty-two, decimal four, five" is written 62.45

"five, decimal eight, three, two" is written _______.832

"three thousand six, decimal zero, four, two" is written

3 ___ ___ 6.0 ___ ___

"seventy, decimal six" is written ___ ___. ___

"three hundred, decimal zero, one, eight, six is written.

____

Self-correcting Exercise #4

Answers may be found on page 49 of this lesson.

1. Read the following decimal numbers.

   (a) 46.05 forty-six, decimal zero, five

   (b) 62 000.41 ______

   (c) 704.068 ______

   (d) 2 000 300.7 ______

2. Use digits to write the following decimal numbers.

   (a) eight million, decimal nine 8 000 000.9

   (b) four, decimal zero, six five, three ______

   (c) one hundred one decimal one, one, four ______

   (d) seventy-three million, decimal one, six ______

E. Expanded Notation Extended

Remember that the expanded form of a number gives the number expressed as a sum. Each term of the sum gives the place value of a digit in the number. The idea of expanded notation can be extended to include decimal numbers.

EXAMPLE: Write 26.307 in expanded form.
Solution

In the numeral 26.307:

- the digit 2 represents 2 tens or (2×10).
- the digit 6 represents __ ones or $(\_\_ \times 1)$.
- the digit 3 represents 3 tenths or $\left(\_\_ \times \frac{1}{10}\right)$.
- the digit 0 represents 0 ______ or $\left(0 \times \frac{1}{100}\right)$.
- the digit 7 represents 7 ______ or $7 \times \_\_$.

Thus, 26.307 can be written in expanded form as follows:

$$26.307 = (2 \times 10) + (6 \times 1) + \left(3 \times \frac{1}{10}\right) + \left(0 \times \frac{1}{100}\right) + \left(7 \times \frac{1}{1000}\right)$$

Self-correcting Exercise #5

Answers may be found on page 49 of this lesson.

1. Write each numeral in expanded form.

   (a) $926.41 = (\_\_ \times \_\_\_\_\_) + (2 \times \_\_\_\_\_) + (\_\_ \times \_\_\_\_\_) + \left(\_\_ \times \frac{1}{10}\right)$

   $+ (\_\_ \times \_\_)$

   (b) $7.832 = (\_\_\_ \times \_\_\_) + (\_\_\_ \times \_\_\_) + (\_\_\_ \times \_\_\_) + (\_\_\_ \times \_\_\_)$

   (c) $20.057 = (\_\_\_ \times \_\_\_) + (\_\_\_ \times \_\_\_) + (\_\_\_ \times \_\_\_) + (\_\_\_ \times \_\_\_)$

   $= (\_\_\_ \times \_\_\_)$

2. Represent each expanded form by a numeral.

   (a) $(7 \times 1) + \left(2 \times \frac{1}{10}\right) =$ ____________

   (b) $(3 \times 10) + (0 \times 1) + \left(5 \times \frac{1}{10}\right) + \left(6 \times \frac{1}{100}\right) =$ ____________

   (c) $(9 \times 100) + (0 \times 10) + (2 \times 1) + \left(0 \times \frac{1}{10}\right) + \left(4 \times \frac{1}{100}\right)$

   $+ \left(6 \times \frac{1}{1000}\right) + \left(2 \times \frac{1}{10\ 000}\right) =$ ____________

F. Ordering Decimal Numbers

Two decimal numbers are equal if they have exactly the same digits in each of the corresponding places. Zeros that appear at the end of a decimal number do not affect the value of the decimal. For example, the following decimals are equal.

$$62.5 = 62.50 = 62.500 = 62.500\ 0 \text{ etc.}$$

In deciding if one decimal is greater than or less than another decimal, we examine the corresponding digits in the decimals much as we do when we order whole numbers. We begin at the digits farthest to the left and compare digits in corresponding places. Move from left to right until you notice two digits in corresponding places that are not the same. The larger of these two digits must occur in the larger number.

EXAMPLE 1: Which is larger; 125.62 or 125.72?

Solution

The digits in the two numbers are the same until you get to the tenths place.

125.62 ← THESE DIGITS ARE DIFFERENT.
125.72

Since the 7 in 125.72 is greater than the 6 in 125.62, 125.72 is the larger number.

i.e. $\underline{125.72 > 125.62}$

EXAMPLE 2: Which is larger; 7.46 or 7.461?

Solution

Rewrite 7.46 as 7.460 so that it will have the same number of decimal places as 7.461. The digits in the two numbers are the same until you get to the thousandths place.

7.460 ← THESE DIGITS ARE DIFFERENT.
7.461

Since the 1 in 7.461 is greater than the 0 in 7.460, 7.461 is the larger number.

i.e. $\underline{7.461 > 7.460}$

Self-correcting Exercise #6

Answers may be found on page 50 of this lesson.

1. Fill in the blanks.

   (a) In the numbers 216.241 and 216.235, the digits are the same until you get to the ______________ place. Since the ____ in 216.241 is greater than the ___ in 216.235, ______ is the larger number.

   ________ > ________

(b) In the numbers 17.104 5 and 17.105, the digits are the same until you get to the ____________ place. Since the 5 in ____________ is greater than the 4 in ____________, ____________ is the larger number.

____________ > ____________

2. State whether the first decimal is <, =, or > the second decimal.

(a) 0.89	_____	0.98	(f) 8.93	_____	8.9300
(b) 0.67	_____	0.670	(g) 5.001	_____	4.999
(c) 0.2	_____	0.199	(h) 0.098 8	_____	0.10
(d) 1.010	_____	0.999	(i) 1.0	_____	1
(e) 0.54	_____	0.45	(j) 0.109	_____	0.10

G. Rounding Numbers

In situations in which extreme accuracy is not required, it is customary to "round" numbers. In rounding whole numbers, we replace some of the actual figures by zeros in order to reduce the number of non-zero digits that we have to work with.

Rounding a whole number involves taking it to a particular place. When rounding is completed only zeros will appear to the right of this place in the number.

Use the following procedure when rounding a whole number to a required number of places.

1. In the number, find the place to which you must round.

2. Look at the digit that occurs to the right of this place. If this digit is less than 5, leave the digit in the place you are rounding to unchanged. If this digit is a 5 or more than 5, increase the digit in the place you are rounding to by one.

3. Insert zeros as placeholders to the right of the place you have rounded.

4. Check your rounded number. Make sure that only zeros appear to the right of the place you have rounded.

EXAMPLE: Round 62,498 to the nearer hundred.

Solution

Find the hundreds place in the numeral.

62 498 hundreds place

Look at the next digit on the right and decide if it is less than 5, or 5 or more.

62 498 hundreds place
next digit on right is more than 5

Increase the digit in the hundreds place by one and replace the digits in the tens and ones places by zeros.

62 500 note that only zeros appear to right of the hundreds place.

Thus, 62 498 rounded to the nearer hundred is 62,500.

We use a similar procedure when rounding decimal numbers to a required number of decimal places. We follow Steps 1 and 2 as outlined on page 15 of this lesson. But we do not add any zeros to the right of the place we have rounded. Zeros at the end of decimal numbers indicate that the number is precise to that number of places. Thus, in a rounded decimal number, no digits should appear to the right of the decimal place to which you have been asked to round.

EXAMPLE: Round 165.234 to the nearer tenth.

Solution

Find the tenths place in the numeral.

165.234 tenths place

Look at the next digit on the right and decide if it is less than 5, or 5 or more.

165.234 tenths place
next digit on right is less than 5.

Leave the digit in the tenths place unchanged. (Zeros are not inserted in the hundredths and thousandths places since this would indicate that rounding had taken place to the nearer thousandth.)

165.2 Notice that no other digits appear to the right of the tenths place.

Thus, 165.234 rounded to the nearer tenth is 165.2.

Self-correcting Exercise #7

Answers may be found on page 51 of this lesson.

1. Fill in the blanks in the chart below. In Step 1, circle the digit that occurs in the place to which you are rounding and put an arrow under the next digit on the right. In Step 2, decide if the digit with the arrow under it is less than 5 or 5 or more. In Step 3, round the given number to the required place by either leaving the circled digit unchanged or increasing it by one. (Insert zeros as placeholders where necessary.)

	Instructions	Step 1	Step 2	Step 3
(a)	Round 9.628 to the nearer hundredth.	9.628	8 is more than 5.	9.63
(b)	Round 173.69 to the nearer ten.	____	________	____
(c)	Round 872.45 to the nearer hundred.	____	________	____
(d)	Round 0.0625 to the nearer thousandth.	____	________	____
(e)	Round 761 847 to the nearer ten thousand.	____	________	____

2. Round each numeral to the required place.

(a) Round 17.032 to the nearer hundredth. __________

(b) Round 179.235 to the nearer ten. __________

(c) Round 63.99 to the nearer tenth. __________

(d) Round 45.000 2 to the nearer thousandth. __________

(e) Round 8053.25 to the nearer hundred. __________

EXERCISE - Decimal Numbers

1. Fill in the blanks.

   (a) The number of digits we have to work with in decimal numeration is __________.

   (b) The number represented by each digit in a numeral is determined by the ____________ the digit occupies in the numeral.

   (c) In the numeral 816 257 003 the digits 816 make up the _______________ period.

   (d) The numeral 285 000 642 032 is read "285 _______________, 642 _______________, 32."

   (e) The only period in a numeral which can contain less than three digits is the period farthest to the _______________.

   (f) Each place in a numeral has a value ten times as great as the next place on the _______________.

   (g) The place that lies one place to the left of the thousands place is the ______________________________ place.

   (h) The names of all the places that lie to the right of the decimal point end in ________.

   (i) The hundreds place lies __________ places to the __________ of the decimal point. The hundredths place lies __________ places to the __________ of the decimal point. (See chart on page 8.)

   (j) 120.263 is read, " ____

   ____."

   (k) In rounding 3.865 to the nearer tenth, we must look at the digit in the ____________________ place and decide if it is less than 5, or 5 or more.

2. Give the missing word and number in each question.

(a) 28.45 The 2 in the tens place means 20.

(b) 28.45 The 4 in the tenths place means $\frac{4}{10}$.

(c) 632.87 The 6 in the ________ place means ____

(d) 632.87 The 7 in the ________ place means ____

(e) 1764.398 The 1 in the ________ place means ____

(f) 1764.398 The 8 in the ________ place means ____

(g) 7.0942 The 7 in the ________ place means ____

(h) 7.0942 The 2 in the ________ place means ____

(i) 32 416.9 The 3 in the ________ place means ____

(j) 32 416.9 The 9 in the ________ place means ____

3. Read each numeral and then write it in expanded form.

(a) 42.609

(i) forty-two, decimal six, zero, nine.

(ii) $(4 \times 10) + (2 \times 1) + (6 \times \frac{1}{10}) + (0 \times \frac{1}{100}) + (9 \times \frac{1}{1000})$

(b) 13.51

(i) ________

(ii) (1 × __) + (__ × __) + (__ × __) + (1 × __)

(c) 1.003 5

(i) ________

(ii) (__ × __) + (0 × __) + (__ × __) + (__ × __) + (__ × __)

(d) 60 100.82

(i) ________

(ii) ________

____

4. Make true statements. Use >, < , or =.

(a) 1.08	________	1.8	(g) 16.1	________	16
(b) 0.35	________	0.39	(h) 1.35	________	1.350 0
(c) 0.4	________	0.40	(i) 0.000 7	________	0.007 0
(d) 3.27	________	3.272	(j) 0.3	________	0.33
(e) 9.90	________	9.09	(k) 4.220 1	________	4.22
(f) 0.42	________	0.402	(l) 13.257	________	13.268

5. For each exercise, give the numbers in order. Begin with the least number. (Review Section F on pages 13 and 14.)

	Order (Least to Greatest)
(a) 2.42, 2.02, 2.22, 2.2	2.02, 2.2, 2.22, 2.42
(b) 0.7, 0.07, 0.077, 0.691	________
(c) 1.3, 1.29, 1.378, 1.377 4	________
(d) 0.044, 0.004 4, 0.44, 0.4	________
(e) 0.110, 0.109, 0.1, 0.112	________

6. Round each numeral to the nearer hundred, ten, tenth, and hundredth.

		Round to nearer			
		hundred	ten	tenth	hundredth
(a)	876.432	900	880	876.4	876.43
(b)	1634.075	________	________	________	________
(c)	189.638 7	________	________	________	________
(d)	108.999	________	________	109.0	________
(e)	545.454	________	________	________	________

7. Write a numeral for each number idea.

(a) eight billion, thirty thousand, seven ______

(b) nine tenths ______

(c) sixty-three million, four hundred thousand ______

(d) seventy, decimal zero, eight ______

(e) $(7 \times 1) + \left(5 \times \frac{1}{10}\right) + \left(3 \times \frac{1}{100}\right)$ ______

(f) $(2 \times 100) + (0 \times 10) + (3 \times 1) + \left(7 \times \frac{1}{10}\right)$ ______

(g) nine thousand seven, decimal six, two, three ______

(h) four thousandths ______

Topic Two: Adding, Subtracting, and Multiplying Decimal Numbers

A. Writing Decimal Numbers as Rational Numbers

Every decimal number is equivalent to a given rational number. We can find this rational number by writing the decimal in expanded form. For example, note that the decimal number 0.129 is equivalent to the rational number $\frac{129}{1000}$.

$$0.129 = \left(1 \times \frac{1}{10}\right) + \left(2 \times \frac{1}{100}\right) + \left(9 \times \frac{1}{1000}\right)$$

$$= \frac{1}{10} + \frac{2}{100} + \frac{9}{1000}$$

Take the three fractions to their L.C.D., 1000.

$$= \frac{100 + 20 + 9}{1000}$$

$$= \frac{129}{1000}$$

This same rational number could be found more readily by writing the digits in the decimal number over the value of the place held by the last digit in the decimal.

i.e. $0.129 = \dfrac{129 \leftarrow \text{THESE ARE THE DIGITS IN THE DECIMAL.}}{1000 \leftarrow \text{THE LAST DIGIT IN THE DECIMAL IS IN THE THOUSANDTHS PLACE.}}$

Once you have found the rational number that is equivalent to a given decimal, always write the rational number in lowest terms. (Recall that this is done by dividing the numerator and denominator by their highest common factor.) For example, note how the decimal 0.16 can be written as a rational number in lowest terms.

$$0.16 = \frac{16 \leftarrow \text{THESE ARE THE DIGITS IN THE DECIMAL.}}{100 \leftarrow \text{THE LAST DIGIT 6 IS IN THE HUNDREDTHS PLACE.}}$$

$$= \left.\frac{16 \div 4}{100 \div 4}\right\} \text{ DIVIDE BOTH TERMS BY THE H.C.F., 4.}$$

$$= \frac{4}{25}$$

Thus, 0.16 is equivalent to the rational number $\frac{4}{25}$.

<u>Self-correcting Exercise #8</u>

Answers may be found on page 51 of this lesson.

1. Fill in the blanks in the chart below. In Step 1, give the value of the place held by the last digit in the numeral. In Step 2, write the digits in the decimal over the value of the place held by the last digit in the decimal. In Step 3, state the H.C.F. of the numerator and denominator. In Step 4, express the rational number in lowest terms.

	Decimal Number	Step 1	Step 2	Step 3	Step 4
(a)	0.001 5	ten-thousandths	$\frac{15}{10\ 000}$	5	$\frac{3}{2000}$
(b)	0.016	________	____	____	____
(c)	0.8	________	____	____	____
(d)	0.26	________	____	____	____
(e)	0.075	________	____	____	____

2. Write each decimal as an equivalent rational number in lowest terms.

(a) $0.45 = \frac{45}{100} = \frac{9}{20}$

(b) $0.006 =$

(c) $0.025 =$

(d) $0.005\,5 =$

(e) $0.2 =$

(f) $0.64 =$

If a decimal numeral contains some non-zero digits to the left of the decimal point, it is equivalent to a mixed number. The digits to the left of the decimal point will be the same as the whole number part of the mixed number. The fraction part can be obtained in the same manner as before. For example,

$$2.08 = 2\frac{8}{100} \longleftarrow \text{.08 IS EQUIVALENT TO } \frac{8}{100}.$$

WHOLE NUMBER PARTS CORRESPOND.

$$= 2\frac{2}{25}$$

Self-correcting Exercise #9

Answers may be found on page 52 of this lesson.

1. Write each decimal as a mixed number in lowest terms.

(a) $3.48 = 3\frac{48}{100} = 3\frac{12}{25}$

(b) $6.032 =$

(c) $10.008\,5 =$

(d) $123.4 =$

(e) $1.128 =$

(f) $7.003 =$

B. Adding and Subtracting Decimal Numbers

Computation with decimals can best be explained if it is related to computation with rational numbers. Note how the following addition problem can be done by changing the decimals to rational numbers and then using the rules for adding rational numbers that we discussed in Lesson 4.

EXAMPLE: $2.4 + 1.3 = \square$

Solution

First, change the decimals to mixed numbers.

$$2.4 + 1.3 = 2\frac{4}{10} + 1\frac{3}{10}$$

Then, write the mixed numbers in rational form.

$$= \frac{24}{10} + \frac{13}{10}$$

Use the rules for adding rational numbers to find the sum.

$$= \frac{24 + 13}{10}$$

$$= \frac{37}{10}$$

$$= 3\frac{7}{10}$$

Convert the answer back to a decimal number.

$$= 3.7$$

Thus, $\underline{\underline{2.4 + 1.3 = 3.7}}$

The same result could be obtained by arranging the two addends in a column so that the decimal points line up, and adding as you would with natural numbers. The decimal point in the sum must be placed directly below the other decimal points.

i.e.
$$\begin{array}{r} 2.4 \\ +1.3 \\ \hline 3.7 \end{array}$$
DECIMAL POINTS LINE UP.

Using this method, add the following groups of numbers.

1. 3.478, 12.961, 100.003

$$\begin{array}{r} 3.478 \\ 12.961 \\ 100.003 \\ \hline \end{array}$$
LINE UP DECIMAL POINTS.

2. 16.65, 123.08, 5.97

← FINISH

The same approach can be used for subtracting decimals. Subtract as if you were working with natural numbers and then insert the decimal point in the answer. (It must line up with the other decimal points.)

EXAMPLE:

$$\begin{array}{r} 8.904 \\ -1.637 \\ \hline 7.267 \end{array}$$

DECIMAL POINTS LINE UP.

Using this method, subtract the following pairs of numbers. (In each case, subtract the second number from the first.)

1. 18.63 9.48	2. 123.7, 16.9	3. 7.001, 5.635
$\begin{array}{r} 18.63 \\ -\ 9.48 \\ \hline \end{array}$		

C. Multiplying Decimal Numbers

Rational numbers can also be used to help explain multiplication of decimals. Study the following example.

EXAMPLE: $2.03 \times 3.7 = \square$

Solution

First, change the decimals to mixed numbers.

$$2.03 \times 3.7 = 2\frac{3}{100} \times 3\frac{7}{10}$$

Then write the mixed number in rational form.

$$= \frac{203}{100} \times \frac{37}{10}$$

Use the rule for multiplying rational numbers to find the product.

$$= \frac{203 \times 37}{100 \times 10}$$

$$= \frac{7511}{1000}$$

$$= 7\frac{511}{1000}$$

Convert the answer back to a decimal number.

$$= 7.511$$

Thus, $\underline{\underline{2.03 \times 3.7 = 7.511}}$

The same result could have been obtained by multiplying the numbers as if they were natural numbers and then locating the decimal point in the product. The number of digits to the right of the decimal point in the product is equal to the sum of the number of digits to the right of the decimal point in each of the two factors. In the example above, the factor 2.03 has two decimal places, the factor 3.7 has one decimal place, and the product 7.511 has two plus one or three decimal places.

$$\text{i.e.}\quad \begin{array}{r l} 2.03 & \leftarrow \text{2 DECIMAL PLACES} \\ \times 3.7 & \leftarrow \text{1 DECIMAL PLACE} \\ \hline 1421 & \\ 609 & \\ \hline 7.511 & \leftarrow \text{(2 + 1) OR 3 DECIMAL PLACES} \end{array}$$

In finding some products, you will have to insert zeros as place-holders in the answer so that you will have the correct number of decimal places.

EXAMPLE: $0.054 \times 0.132 =$ ▨

Solution

Since both factors have 3 decimal places, the product must have (3 + 3) or 6 decimal places.

$$\begin{array}{r} 0.132 \\ \times 0.054 \\ \hline 528 \\ 660 \\ \hline .007128 \end{array}$$

TWO ZEROS MUST BE INSERTED IN ORDER TO OBTAIN 6 DECIMAL PLACES.

Self-correcting Exercise #10

Answers may be found on page 52 of this lesson.

1. Find how many decimal places there will be in each of the following products.

   (a) $6.83 \times 5.07 =$ ▨

   The product has (2 + 2) or 4 decimal places.

   NUMBER OF DECIMAL PLACES IN 6.83

   NUMBER OF DECIMAL PLACES IN 5.07

   (b) $3.004 \times 7.1 =$ ▨

   The product has (__ + __) or __ decimal places.

   (c) $8 \times 33.406\ 2 =$ ▨

   The product has (__ + __) or __ decimal places.

   (d) $23.025 \times 16.14 =$ ▨

   The product has (__ + __) or __ decimal places.

2. Locate the decimal point in each of the following products. (Insert zeros in the products where necessary.)

   (a) $2.5 \times 13.5 = 3375$

   (b) $0.25 \times 1.35 = 3375$

   (c) $25 \times 13.5 = 3375$

   (d) $0.025 \times 0.135 = 3375$

   (e) $2.5 \times 1.35 = 3375$

   (f) $0.25 \times 0.135 = 3375$

3. Find these products.

   (a) $\begin{array}{r} 0.35 \\ \times 0.6 \\ \hline \end{array}$

   (b) $\begin{array}{r} 6.25 \\ \times 1.78 \\ \hline \end{array}$

Now, find the following products on your own.

1. $\begin{array}{r} 0.042 \\ \underline{\times 7.9} \end{array}$ 2. $\begin{array}{r} 2.004 \\ \underline{\times 8} \end{array}$ 3. $\begin{array}{r} 6.018 \\ \underline{\times 0.25} \end{array}$

D. Multiplying by Powers of 10

Powers of 10 are numbers like 10, 100, 1000, and so on.

We can use short-cuts when we multiply decimal numbers by powers of 10. Study the following multiplication question.

EXAMPLE:

$\begin{array}{r} 0.17 \\ \underline{\times 100} \\ 17.00 \end{array}$

Notice the relationship between the factor 0.17 and the product 17. The digits in these two numbers are the same, but the decimal point has been moved two places to the right in the product. The number of places the decimal point has moved corresponds to the number of zeros in the other factor, 100.

> In general, to multiply a decimal by any power of 10, move the decimal point the same number of places to the right as there are zeros in the multiplier.

For example,

$7.8364 \times 1000 = 7836.4$ } Since there are 3 zeros in 1000, the decimal point has been moved 3 places to the right.

If you were multiplying by 100 000 how many places to the right would you have to move the decimal point? ____________

In some cases, you will have to insert zeros as placeholders so that the decimal point can be moved the correct number of places.

For example,

$2.35 \times 10\ 000 = 23\ 500$ } Two zeros had to be inserted so that the decimal point could be moved 4 places to the right.

If you multiplied 1.36 by 1000, how many zeros would you have to insert as placeholders in the product? ______________________

Self-correcting Exercise #11

Answers may be found on page 53 of this lesson.

1. Tell how many places to the right you will have to move the decimal point in the first factor in order to find the product.

   (a) $0.052 \times 10 =$ ▨

   The decimal point must be moved _____ place(s) to the right.

   (b) $79.862 \times 10\ 000 =$ ▨

   The decimal point must be moved _____ place(s) to the right.

   (c) $4.369\ 2 \times 100 =$ ▨

   The decimal point must be moved _____ place(s) to the right.

   (d) $0.039 \times 1000 =$ ▨

   The decimal point must be moved _____ place(s) to the right.

   (e) $3.2 \times 1\ 000\ 000 =$ ▨

   The decimal point must be moved _____ place(s) to the right.

2. Multiply each number by 10, 100, and 1000.

		Multiply by		
	Number	10	100	1000
(a)	3.02	30.2	302	3020
(b)	7.9	______	______	______
(c)	0.005	______	______	______
(d)	0.36	______	______	______
(e)	0.146 7	______	______	______

3. Fill in the blanks.

(a) $10 \times 72.653 =$ ______

(b) $100 \times 14 =$ ______

(c) $1000 \times 23.4 =$ ______

(d) $0.036 \times 10\ 000 =$ ______

Now, find the following products on your own.

1. $83.5 \times 1000 =$ ______

2. $100 \times 0.9 =$ ______

3. $0.058 \times 10\ 000 =$ ______

4. $10 \times 467 =$ ______

5. $1000 \times 0.03 =$ ______

6. $6.432 \times 100 =$ ______

7. $10 \times 0.002 =$ ______

8. $1000 \times 32 =$ ______

9. $100 \times 62.4 =$ ______

10. $10 \times 0.423 =$ ______

11. $10.2 \times 1000 =$ ______

12. $0.000\ 32 \times 100 =$ ______

Topic Three: Dividing Decimal Numbers

Division is an operation that involves the separation of one large group into a number of small groups. For example, in the division question $16 \div 3 = \square$, we wish to divide a group of 16 objects into smaller groups containing 3 objects each.

A group of 16 objects contains 5 smaller groups of 3 objects with one object remaining.

$$16 \div 3 = 5, \text{ remainder } 1$$

We have special names for each part of a division problem. The number you divide by is called the DIVISOR; the number you divide into is called the DIVIDEND; the answer you obtain is called the QUOTIENT; and the number left over is called the REMAINDER. Learn these terms, and know what parts of a division question they refer to.

$$\underset{\text{dividend}}{16} \div \underset{\text{divisor}}{3} = \underset{\text{quotient}}{5}, \text{ remainder } 1$$

A. Dividing Whole Numbers

In the elementary grades, many students have been taught a method of division in which they subtract multiples of the divisor from the dividend until they obtain a remainder that is less than the divisor. The number of groups that have been taken out are then added to give the quotient. Since multiples of the divisor may be subtracted in many different ways, the correct solution to a division question may appear in a variety of forms. For example, note the three correct solutions to the division problem given at the top of page 34.

$$937 \div 18 = \boxed{}$$

```
18 | 937 | 20          18 | 937 | 40          18 | 937 | 50
     360                     720                     900
     ---                     ---                     ---
     577 | 20                217 | 10                 37 |  2
     360                     180                      36
     ---                     ---                     ---------
     217 | 10                 37 |  2                  1 | 52
     180                      36                       ↑    ↑
     ---                     ---------           Remainder  Quotient
      37 |  2                  1 | 52
      36                       ↑    ↑
     ---------           Remainder  Quotient
       1 | 52
       ↑    ↑
Remainder   Quotient (number of
            times 18 has been
            subtracted from 937.)
```

In all three cases, we find that 937 ÷ 18 = 52, remainder 1.

Although this method of division gives the student a good understanding of the division process, it is not the most efficient method of division available. Also, difficulties arise when this method is used to divide decimal numbers. For these reasons, all high school students are urged to use the method of division discussed below.

EXAMPLE: Divide 937 by 18.

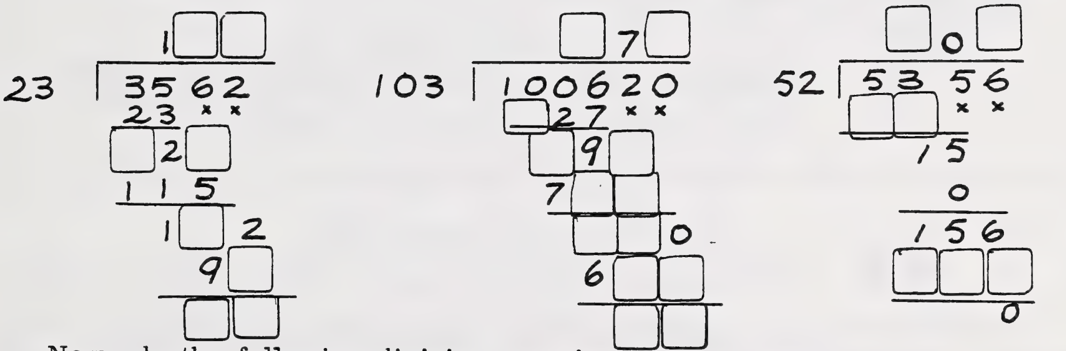
$\begin{array}{r} 5 \\ 18\,\overline{)937} \end{array}$	18 will not divide into 9, so we must begin by dividing 18 into 93. 18 divides into 93 about 5 times. Place this partial quotient 5 above the <u>last</u> digit in 93.
$\begin{array}{r} 5 \\ 18\,\overline{)937} \\ \underline{90} \\ 3 \end{array}$	Multiply 18 by 5 and place the product 90 below 93. Subtract 90 from 93 and obtain a remainder of 3.
$\begin{array}{r} 5 \\ 18\,\overline{)937} \\ \underline{90}^{\times} \\ 37 \end{array}$	Bring down the next digit, 7, in the dividend and place it to the right of the remainder, 3. (The small × under the 7 indicates that it has been brought down.)

52 18 ⟌ 937 90 37	Divide 18 into 37. It goes about twice. Place this partial quotient, 2, beside the first partial quotient, 5. (i.e. the quotient 52 appears above the 37 in the dividend.)
52 18 ⟌ 937 90 37 36 1	Multiply 18 by 2 and place the product 36 below 37. Subtract 36 from 37 and obtain a remainder of 1. (Again, check to make sure that your remainder is less than the divisor.) The division is complete since there are no more digits to bring down. (If there were, you would bring down these digits one at a time and continue in the same manner.)
	937 ÷ 18 = 52, remainder 1

Self-correcting Exercise #12

Answers may be found on page 54 of this lesson.

1. Work through each division problem below, filling in the boxes as you go along.



Now, do the following division questions on your own.

32 ⟌ 3392   121 ⟌ 42350   20 ⟌ 36547

B. <u>Decimal Dividends</u>

If the dividend (number you are dividing into) is a decimal rather than a whole number, your answer will also be a decimal. The decimal point in the answer will appear <u>directly above</u> the decimal point in the dividend. Once you have placed the decimal point in the answer you can go ahead and divide as before.

EXAMPLE: Divide 161.92 by 32.

. 32)161.92 DECIMAL POINTS LINE UP	Begin by placing the decimal point where it will appear in the answer.
5. 32)161.92 160 1	32 divides into 161 about 5 times. Place this partial quotient 5 above the <u>last</u> digit in 161. (It is <u>very</u> important that the 5 is placed correctly. Otherwise, the decimal point will be in the wrong place in the answer.) Multiply 32 by 5 and subtract this product from 161.
5.0 32)161.92 160 19 0 19	Bring down the 9. 32 divides into 19 zero times. Place this partial quotient 0 above the 9 in the dividend. Multiply 0 and 32 to get 0. Subtract 0 from 19.
5.06 32)161.92 160 19 0 192 192 0	Bring down the 2. 32 divides into 192 6 times. Place this partial quotient 6 above the last digit in the dividend. Multiply 6 and 32 and subtract this product from 192. The division is complete. The quotient is 5.06. (Note that the quotient has the same number of decimal places as the dividend.)
	161.92 ÷ 32 = 5.06

In some division questions, you will have to insert zeros to act as placeholders between the decimal point and the first digit in the quotient.

EXAMPLE: Divide 1.28 by 320.

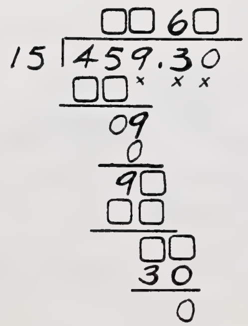
$\begin{array}{r} . \\ 320 \overline{)1.28} \end{array}$ LINE UP DECIMAL POINTS.	Place the decimal point in the answer above the decimal point in the dividend. Note that 320 will not divide into 128.
$\begin{array}{r} .\ \ 4 \\ 320 \overline{)1.280} \end{array}$	Add one zero to 1.28. Divide 320 into 1280. It goes 4 times. Place the quotient 4 above the <u>last</u> digit in 1280.
$\begin{array}{r} .\ \ 4 \\ 320 \overline{)1.280} \\ \underline{1\ 280} \\ 0 \end{array}$	Multiply 4 and 320 and subtract this product from 1280.
$\begin{array}{r} .004 \\ 320 \overline{)1.280} \\ \underline{1\ 280} \\ 0 \end{array}$	Insert two zeros between the decimal point and the quotient, 4. These zeros represent tenths and hundredths.
$1.28 \div 320 = .004$	

<u>Self-correcting Exercise #13</u>

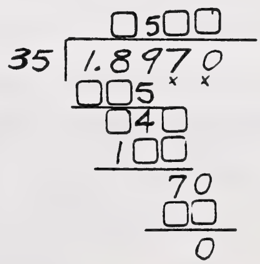
Answers may be found on page 54 of this lesson.

Begin by placing the decimal point in the quotient in its proper position above the decimal point in the dividend. Then, work through each division question, filling in the blanks as you go along.

1.



2.



Now do the following division questions on your own. Make sure that the decimal point is placed correctly in your answer.

$136 \overline{)7.072}$

$21 \overline{)253.89}$

$230 \overline{)11.50}$

$234 \overline{)0.7254}$

C. <u>Decimal Divisors</u>

If the divisor (number you are dividing by) is a decimal rather than a natural number, you will have to change it to a natural number before you can divide. Since both the divisor and dividend can be multiplied by any number except zero without changing the quotient, this adjustment can be made by multiplying both the divisor and dividend by the appropriate power of 10. A mark called a caret (^) is used to indicate the new position of the decimal point in the two numbers.

For example, suppose you were asked to divide 0.2415 by 0.035.

In order to make the divisor a natural number, we must multiply both numbers by 1000. In other words, the decimal point must be moved three places to the right in both the divisor and dividend. We can indicate the new positions of the decimal point by inserting a caret after the 5 in the divisor and after the 1 in the dividend. The decimal point in the answer will appear above the caret in the dividend.

$$0.035_{\wedge}\overline{)\,0.241_{\wedge}5}$$

POSITION OF DECIMAL POINT IN THE ANSWER

Once you have made the divisor a natural number, you are faced with the task of dividing a decimal by a natural number. You were shown how to do this in Part B on pages 34 and 35 of this lesson.

$$\begin{array}{r} 6.9 \\ 0.035_{\wedge}\overline{)\,0.241_{\wedge}5} \\ \underline{210} \\ 315 \\ \underline{315} \\ 0 \end{array}$$

Begin by dividing 35 into 241. Place the quotient, 6, above the last digit in 241.

Thus, 0.2415 ÷ 0.035 = __________.

EXAMPLE: Divide 7.5 by 0.25.

$0.25_{\wedge}\overline{)\,7.50_{\wedge}}$	To make the divisor a whole number, you must move the decimal point two places to the right. Place a caret after the 5 in the divisor. In order to move the decimal point two places to the right in the dividend, you must add a zero. Place a caret after the 0 in the dividend. Insert a decimal point above the caret in the dividend.
$\begin{array}{r} 3\,. \\ 0.25_{\wedge}\overline{)\,7.50_{\wedge}} \\ \underline{7\,5} \\ 0 \end{array}$	25 goes into 75 three times, with no remainder. Place the quotient 3 above the last digit in 75. Multiply 3 and 25 to obtain a product of 75. Division is complete since the remainder is 0 and there is only a 0 left to bring down.
$\begin{array}{r} 30. \\ 0.25_{\wedge}\overline{)\,7.50_{\wedge}} \\ \underline{7\,5} \\ 0 \end{array}$	Insert a zero after the 3 to hold the place of the ones.
7.5 ÷ 0.25 = 30	

Self-correcting Exercise #14

Answers may be found on page 54 of this lesson.

1. In each question, change the divisor to a natural number. Use carets in the divisor and dividend to indicate the new position of the decimal point. Place a decimal point above the caret in the dividend to show where the decimal point will be positioned in the answer. (Do not complete the division.)

(a) $.06 \overline{)0.452}$ (b) $0.931 \overline{)6.2}$

(c) $0.5 \overline{)0.631}$ (d) $0.031 \overline{)92}$

(e) $3.15 \overline{)0.763}$ (f) $42.6 \overline{)50}$

2. In each question, change the divisor to a natural number. (Use carets in the divisor and dividend.) Place the decimal point in its proper position in the answer. Then, go ahead and divide.

(a) $0.136 \overline{)51}$

(b) $34.1 \overline{)695.64}$

Now, do the following division questions on your own.

1. $0.13 \overline{)52}$

2. $1.5 \overline{)0.4005}$

3. $0.315 \overline{)1.9845}$

4. $0.93 \overline{)0.05208}$

D. Finding Quotients With a Required Number of Decimal Places

Some division questions do not work out evenly. In such cases, you are often asked to round the quotient to a required number of decimal places. This can be done only if this same number of decimal places also appears in the dividend. Zeros must be added to the dividend until this is the case. The final remainder obtained can be used in rounding the quotient. Look at the remainder and compare it with the divisor. If the remainder is less than half the divisor, leave the final digit in the quotient unchanged. If the remainder is more than half the divisor (or exactly half the divisor), increase the last digit in the quotient by 1.

EXAMPLE: Divide 0.55 by 3.2. Round your answer to <u>3 decimal places</u>.

$3.2_{\wedge}\overline{)\,0.5_{\wedge}5}$	Begin by moving the decimal point one place to the right in both the divisor and dividend. Also, position the decimal point above the caret in the dividend.
$3.2_{\wedge}\overline{)\,0.5_{\wedge}500}$	Since you want 3 decimal places in your answer, you must also have 3 decimal places in the dividend. As it stands, 5.5 has only 1 decimal place. Thus, you must insert 2 zeros to the right of 5.5
$\begin{array}{r} .171 \\ 3.2_{\wedge}\overline{)\,0.5_{\wedge}500} \\ \underline{32} \\ 230 \\ \underline{224} \\ 60 \\ \underline{32} \\ 28 \end{array}$	Divide as before.
The quotient rounded to 3 decimal places is <u>0.172</u>	In order to round the quotient to 3 decimal places, look at the final remainder, 28. Since it is more than half the divisor, 32, increase the last digit in the quotient by 1. (i.e. 0.171 becomes 0.172.)
$0.55 \div 3.2 = 0.172$ (to 3 decimal places)	

Self-correcting Exercise #15

Answers may be found on page 55 of this lesson.

1. Round the answer to 3 decimal places.

   $6.1 \overline{)\,0.095}$

   Rounded quotient is ________.

2. Round the answer to 1 decimal place.

   $0.52 \overline{)\,317}$

   Rounded quotient is ________.

Now, do the following division questions on your own.

1. Round each answer to 3 decimal places.

   (a)

   $4.2 \overline{)\,67.3}$

   Rounded quotient is ________.

   (b)

   $8.4 \overline{)\,0.238}$

   Rounded quotient is ________.

2. Round each answer to 1 decimal place.

(a)

$0.36\overline{)21.7}$

(b)

$5.1\overline{)32.58}$

Rounded quotient is ________.

Rounded quotient is ________.

E. Dividing by Powers of 10

On page 30 of this lesson, you were given some short-cuts for multiplying numbers by powers of 10. Now let us look for some short-cuts for dividing numbers by powers of 10.

Study the following division problem.

EXAMPLE: $36.54 \div 1000 = \boxed{}$

Solution

$$
\begin{array}{r}
.03654 \\
1000\overline{)36.54000} \\
\underline{30\ 00} \\
6\ 540 \\
\underline{6\ 000} \\
5400 \\
\underline{5000} \\
4000 \\
\underline{4000} \\
0
\end{array}
$$

Notice the relationship between the dividend 36.54 and the quotient 0.03654. The digits in these two numbers are the same, but the decimal point has been moved <u>three</u> places to the <u>left</u> in the quotient. The number of places the decimal point has been moved corresponds to the number of zeros in the divisor, 1000.

In general, to divide a decimal by any power of 10, move the decimal point the same number of places to the <u>left</u> as there are zeros in the divisor.

For example,

$238.4 \div 100 = 2.384$ — Since there are 2 zeros in 100, the decimal point has been moved 2 places to the left.

If you were dividing by 10 000, how many places to the left would you have to move the decimal point? ______________________________

In some cases, you will have to insert zeros as placeholders so that the decimal point can be moved the correct number of places.

For example,

$1.5 \div 1000 = .0015$ — Two zeros had to be inserted so that the decimal point could be moved 3 places to the left.

If you divided 263.5 by 10 000, how many zeros would you have to insert as placeholders in the quotient? ______________________________

<u>Self-correcting Exercise #16</u>

Answers may be found on page 56 of this lesson.

1. Tell how many places to the left you will have to move the decimal point in the dividend in order to find the quotient.

(a) $31.5 \div 10 = \boxed{}$

The decimal point must be moved _____ place(s) to the left.

(b) $368.427 \div 10\ 000 = \boxed{}$

The decimal point must be moved _____ place(s) to the left.

(c) $0.043 \div 100 = \boxed{}$

The decimal point must be moved _____ place(s) to the left.

(d) $2.638 \div 1000 = \boxed{}$

The decimal point must be moved _____ place(s) to the left.

(e) $83\ 674.2 \div 1\ 000\ 000 = \boxed{}$

The decimal point must be moved _____ place(s) to the left.

2. Divide each number by 10, 100, and 1000.

Number	Divide by 10	Divide by 100	Divide by 1000
(a) 62.5	6.25	0.625	0.0625
(b) 837.46	______	______	______
(c) 6.9	______	______	______
(d) 16 439	______	______	______
(e) 0.01	______	______	______

3. Fill in the blanks.

(a) $82.74 \div 100 =$ ______
(b) $33.52 \div 1000 =$ ______
(c) $0.05 \div 10 =$ ______
(d) $6843.2 \div 10\ 000 =$ ______
(e) $23.4 \times 100 =$ ______
(f) $6.9 \times 1000 =$ ______
(g) $0.07 \div 1000 =$ ______
(h) $2.115 \times 10 =$ ______

EXERCISE - Operating with Decimal Numbers

1. Fill in the blanks.

(a) In a product of decimal numbers, the number of digits to the right of the decimal point is equal to the ______________ of the number of digits to the ______________ of the decimal point in each factor.

(b) When dividing two decimal numbers, we must change the ______________ to a natural number before we can divide. A mark called a ______________ is used to indicate the new position of the decimal point. The decimal point is moved the same number of places to the ________ in both the divisor and ______________. The decimal point in the quotient must appear above the caret in the ______________.

(c) When multiplying a decimal number by 10 000, we must move the decimal point ___ places to the ______________.

(d) When dividing a decimal number by 1 000, we must move the decimal point ___ places to the ______________.

(e) In a division question, the answer you obtain is called the ______________________.

(f) In a multiplication question, the answer you obtain is called the ____________________.

2. Write each decimal as a fraction.

(a) 0.7 = __________ (b) 0.29 = __________

(c) 0.001 = __________ (d) 0.001 7 = __________

3. Write each decimal as a mixed number.

(a) 12.1 = $12\frac{1}{10}$ (b) 132.67 = __________

(c) 6.99 = __________ (d) 100.03 = __________

4. Add the following sets of numbers.

(a) 62.49, 894.71, 123.62, 673.00, 86.43

(b) 94.3, 86.7 125.2, 13.6, 19.9

(c) 132.456, 2.015, 37.916, 0.070

(d) 3.015 6, 0.103 7, 0.006 9, 0.000 4

5. Subtract.

(a) 0.250 from 5.709

(b) 17.043 6 from 27.978 5

(c) 29.18 from 30.63

(d) 1.305 67 from 7.283 64

6. Multiply.

(a)
$$\begin{array}{r} 32 \\ \underline{\times 0.04} \end{array}$$

(b)
$$\begin{array}{r} 17.23 \\ \underline{\times 10.9} \end{array}$$

(c)
$$\begin{array}{r} 1.057 \\ \underline{\times 0.035} \end{array}$$

7. Divide until you obtain a remainder of zero.

(a) $13\overline{)0.585}$

(b) $36\overline{)154.8}$

(c) $42.1\overline{)44.626}$

(d) $0.362\overline{)195.48}$

(e) $0.062\overline{)0.0093}$

(f) $0.12\overline{)0.001836}$

8. Divide. Round each answer to 1 decimal place.

(a) $0.073 \overline{)\ 7}$

(b) $21.3 \overline{)\ 69.5}$

Rounded quotient is ________. Rounded quotient is ________.

9. Divide. Round each answer to 3 decimal places.

(a) $2.4 \overline{)\ 12.98}$

(b) $6.8 \overline{)\ 0.09}$

Rounded quotient is ________. Rounded quotient is ________.

10. Fill in the blanks.

(a) $10 \times 0.27 =$ ________

(b) $100 \times 1.4 =$ ________

(c) $4.213 \div 100 =$ ________

(d) $0.13 \div 1000 =$ ________

(e) $100\ 000 \times 0.007\ 1 =$ ________

(f) $43.7 \div 10\ 000 =$ ________

(g) $0.4 \div 100 =$ ________

(h) $10 \times 34 =$ ________

(i) $1000 \times 28.9 =$ ________

(j) $0.3 \times 100 =$ ________

(k) $1.71 \div 1000 =$ ________

(l) $2.4 \div 10 =$ ________

Key to Self-correcting Exercises in this Lesson

Exercise #1, page 5

1. (a) (i) The digits 68 are in the millions period.
(ii) The digits 32 are in the last two places of the units period.
(iii) The number written in digits is 68 000 032.
Zeros have been used as placeholders in the entire thousands period and in the hundreds place.

(b) (i) The digit 5 is in the billions period.
(ii) The digits 56 are in the last two places of the millions period.
(iii) The digit 6 is in the last place of the thousands period.
(iv) The digits 125 are in the units period.
(v) The number written in digits is 5 056 006 125.
Zeros have been used as placeholders in the hundred-millions, the hundred-thousands place and the ten-thousands place.

2. (a) 83 609 050 (b) 60 004 200 069 (c) 2 008 000

(d) 900 045 003 (e) 6 000 837 000

Exercise #2, page 7

1. (a) $1\ 560 = (1 \times 1000) + (5 \times 100) + (6 \times 10) + (0 \times 1)$

(b) $164\ 397 = (1 \times 100\ 000) + (6 \times 10\ 000) + (4 \times 1000) + (3 \times 100) + (9 \times 10) + (7 \times 1)$

(c) $3\ 067\ 259 = (3 \times 1\ 000\ 000) + (0 \times 100\ 000) + (6 \times 10\ 000) + (7 \times 1000) + (2 \times 100) + (5 \times 10) + (9 \times 1)$

2. (a) 430 520 (b) 8 050 333 (c) 55 100 205

FIRST DIGIT IS IN THE HUNDRED-THOUSANDS PLACE.

Exercise #3, page 10

1. (a) 1 thousand = 10 hundreds (Hundreds lie to the right of thousands.)

(b) 1 tenth = 10 hundredths (Hundredths lie to the right of tenths.)

(c) 1 thousandth = 10 ten-thousandths (Ten-thousandths lie to the right of thousandths.)

(d) 1 hundred = $\frac{1}{10}$ of 1 thousand (Thousands lie to the left of hundreds.)

(e) 1 ten = $\frac{1}{10}$ of 1 hundred (Hundreds lie to the left of tens.)

(f) 1 thousandth = $\frac{1}{10}$ of 1 hundredth (Hundredths lie to the left of thousandths.)

(g) 10 ones = 1 ten (Ones lie to the right of tens.)

(h) 1 one = 10 tenths (Tenths lie to the right of ones.)

2. (a) The digit 9 is in the tenths place and represents $\frac{9}{10}$.

(b) The digit 4 is in the thousandths place and represents $\frac{4}{1000}$.

(c) The digit 5 is in the hundredths place and represents $\frac{5}{100}$.

(d) The digit 6 is in the tenths place and represents $\frac{6}{10}$.

(e) The digit 7 is in the ten-thousandths place and represents $\frac{7}{10\ 000}$.

Exercise #4, page 12

1. (a) 46.05 forty-six, decimal zero, five
   (b) 62 000.41 sixty-two thousand, decimal four, one
   (c) 704.068 seven hundred four, decimal zero, six, eight
   (d) 2 000 300.7 two million, three hundred, decimal seven.

2. (a) 8 000 000.9 (b) 4.0653
   (c) 101.114 (d) 73 000 000.16

Exercise #5, page 13

1. (a) $926.41 = (9 \times 100) + (2 \times 10) + (6 \times 1) + \left(4 \times \frac{1}{10}\right) + \left(1 \times \frac{1}{100}\right)$

(b) $7.832 = (7 \times 1) + \left(8 \times \frac{1}{10}\right) + \left(3 \times \frac{1}{100}\right) + \left(2 \times \frac{1}{1000}\right)$

(c) $20.057 = (2 \times 10) + (0 \times 1) + \left(0 \times \frac{1}{10}\right) + \left(5 \times \frac{1}{100}\right) + \left(7 \times \frac{1}{1000}\right)$

2. (a) 7.2 (b) 30.56 (c) 902.046 2

Exercise #6, page 14

1. (a) In the numbers 216.241 and 216.235, the digits are the same until you get to the hundredths place. Since the 4 in 216.241 is greater than the 3 in 216.235, 216.241 is the larger number.

$$216.241 > 216.235$$

(b) In the numbers 17.104 5 and 17.105, the digits are the same until you get to the thousandths place. Since the 5 in 17.105 is greater than the 4 in 17.104 5, 17.105 is the greater number.

$$17.105 > 17.104\ 5$$

2. (a) 0.89 $<$ 0.98 (Look at the digits in the tenths place.)

(b) 0.67 $=$ 0.670 (A number is unaffected by zeros added to the right of the decimal part.)

(c) 0.2 $>$ 0.199 (Look at the digits in the tenths place.)

(d) 1.010 $>$ 0.999 (Look at the digits in the ones place.)

(e) 0.54 $>$ 0.45 (Look at the digits in the tenths place.)

(f) 8.93 $=$ 8.930 0 (Zeros have been added after the decimal part.)

(g) 5.001 $>$ 4.999 (Look at the digits in the ones place.)

(h) 0.098 8 $<$ 0.10 (Look at the digits in the tenths place.)

(i) 1.0 $=$ 1 (Zero has been added to the right of the decimal point.)

(j) 0.109 $>$ 0.100 (Rewrite 0.10 as 0.100 and then look at the digits in the thousandths place.)

Exercise #7, page 17

1.	(a) Round 9.628 to the nearer hundredth.	9.628	8 is more than 5.	9.63
	(b) Round 173.69 to the nearer ten.	173.69	3 is less than 5.	170
	(c) Round 872.45 to the nearer hundred.	872.45	7 is more than 5.	900
	(d) Round 0.0625 to the nearer thousandth.	0.0625	5 equals 5.	0.063
	(e) Round 761 847 to the nearer ten thousand.	761 847	1 is less than 5.	760 000

2. (a) 17.03 (Leave the digit in the hundredths place unchanged.)

   (b) 180 (Increase the digit in the tens place by one.)

   (c) 64.0 (Increase digit in the tenths place by one. This forces you to also increase the digit in the units place by one.)

   THIS MUST BE INCLUDED TO SHOW WHAT PLACE YOU HAVE ROUNDED TO.

   (d) 45.000 (Leave the digit in the thousandths place unchanged.)

   (e) 8100 (Increase the digit in the hundreds place by one.)

Exercise #8, page 22

1.	(a) 0.001 5	ten-thousandths	$\frac{15}{10\ 000}$	5	$\frac{3}{2000}$
	(b) 0.016	thousandths	$\frac{16}{1000}$	8	$\frac{2}{125}$
	(c) 0.8	tenths	$\frac{8}{10}$	2	$\frac{4}{5}$
	(d) 0.26	hundredths	$\frac{26}{100}$	2	$\frac{13}{50}$
	(e) 0.075	thousandths	$\frac{75}{1000}$	25	$\frac{3}{40}$

2. (a) $0.45 = \frac{45}{100} = \frac{9}{20}$

(b) $0.006 = \frac{6}{1000} = \frac{3}{500}$

(c) $0.025 = \frac{25}{1000} = \frac{1}{40}$

(d) $0.0055 = \frac{55}{10\ 000} = \frac{11}{2000}$

(e) $0.2 = \frac{2}{10} = \frac{1}{5}$

(f) $0.64 = \frac{64}{100} = \frac{16}{25}$

Exercise #9, page 23

1. (a) $3.48 = 3\frac{48}{100} = 3\frac{12}{25}$

(b) $6.032 = 6\frac{32}{1000} = 6\frac{4}{125}$

(c) $10.0085 = 10\frac{85}{10\ 000} = 10\frac{17}{2000}$

(d) $123.4 = 123\frac{4}{10} = 123\frac{2}{5}$

(e) $1.128 = 1\frac{128}{1000} = 1\frac{16}{125}$

(f) $7.003 = 7\frac{3}{1000}$

Exercise #10, page 27

1. (a) Since 6.83 has 2 decimal places and 5.07 has 2 decimal places, the product has (2 + 2) or 4 decimal places.

   (b) Since 3.004 has 3 decimal places and 7.1 has 1 decimal place, the product has (3 + 1) or 4 decimal places.

   (c) Since 8 has 0 decimal places and 33.4062 has 4 decimal places, the product has (0 + 4) or 4 decimal places.

   (d) Since 23.025 has 3 decimal places and 16.14 has 2 decimal places, the product has (3 + 2) or 5 decimal places.

2. (a) $2.5 \times 13.5 = 33.75$ (1 + 1 = 2 decimal places in answer)

   (b) $0.25 \times 1.35 = 0.337\ 5$ (2 + 2 = 4 decimal places in answer)

   A ZERO PLACED IN FRONT OF THE DECIMAL POINT HELPS MAKE IT STAND OUT.

   (c) $25 \times 13.5 = 337.5$ (0 + 1 = 1 decimal place in answer)

   (d) $0.025 \times 0.135 = 0.003\ 375$ (3 + 3 = 6 decimal places in answer)

   (e) $2.5 \times 1.35 = 3.375$ (1 + 2 = 3 decimal places in answer)

   (f) $0.25 \times 0.135 = 0.033\ 75$ (2 + 3 = 5 decimal places in answer)

3. (a)
$$\begin{array}{r} 0.35 \\ \times 0.6 \\ \hline 0.210 \end{array}$$

2 + 1 = 3 DECIMAL PLACES IN ANSWER

(b)
$$\begin{array}{r} 6.25 \\ \times 1.78 \\ \hline 5000 \\ 4375 \\ 625 \\ \hline 11.\underbrace{1250}_{2+2\ =\ 4\ \text{DECIMAL PLACES IN ANSWER}} \end{array}$$

<u>Exercise #11, page 29</u>

1. (a) The decimal place must be moved <u>1</u> place to the right (since 10 has 1 zero).

   (b) The decimal point must be moved <u>4</u> places to the right (since 10 000 has 4 zeros).

   (c) The decimal point must be moved <u>2</u> places to the right (since 100 has 2 zeros).

   (d) The decimal point must be moved <u>3</u> places to the right (since 1000 has 3 zeros).

   (e) The decimal point must be moved <u>6</u> places to the right (since 1 000 000 has 6 zeros).

2.

Number	Multiply by		
	10	100	1000
(a) 3.02	30.2	302	3020
(b) 7.9	79	790	7900
(c) 0.005	0.05	0.5	5
(d) 0.36	3.6	36	360
(e) 0.146 7	1.467	14.67	146.7

3. (a) $10 \times 72.653 = \underline{726.53}$ (Move decimal point in 72.653 <u>one</u> place to the <u>right</u>.)

   (b) $100 \times 14 = \underline{1400}$ (Move decimal point in 14 <u>two</u> places to the <u>right</u>. Two zeros must be added as placeholders.)

   (c) $1000 \times 23.4 = \underline{23,400}$ (Move decimal point in 23.4 <u>three</u> places to the <u>right</u>. Two zeros must be added as placeholders.)

   (d) $0.036 \times 10\ 000 = \underline{360}$ (Move decimal point in 0.036 <u>four</u> places to the <u>right</u>. One zero must be added as a placeholder.)

Exercise #12, page 33

1. (a)
$$\begin{array}{r} 154 \\ 23\overline{)3562} \\ \underline{23} \\ 126 \\ \underline{115} \\ 112 \\ \underline{92} \\ 20 \end{array}$$

(b)
$$\begin{array}{r} 976 \\ 103\overline{)100620} \\ \underline{927} \\ 792 \\ \underline{721} \\ 710 \\ \underline{618} \\ 92 \end{array}$$

(c)
$$\begin{array}{r} 103 \\ 52\overline{)5356} \\ \underline{52} \\ 15 \\ \underline{0} \\ 156 \\ \underline{156} \\ 0 \end{array}$$

Exercise #13, page 35

1. (a)
$$\begin{array}{r} 30.62 \\ 15\overline{)459.30} \\ \underline{45} \\ 09 \\ \underline{0} \\ 9\,3 \\ \underline{9\,0} \\ 30 \\ \underline{30} \\ 0 \end{array}$$

(b)
$$\begin{array}{r} .0542 \\ 35\overline{)1.8970} \\ \underline{1\,75} \\ 147 \\ \underline{140} \\ 70 \\ \underline{70} \\ 0 \end{array}$$

Exercise #14, page 38

1. (a) Move both decimal points 2 places to the right. (See carets.) Place decimal above caret in dividend.

$.06_\wedge\overline{).45_\wedge2}$

(b) Move both decimal points 3 places to the right.

$0.931_\wedge\overline{)6.200_\wedge}$

TWO ZEROS MUST BE ADDED

(c) Move both decimal points 1 place to the right.

$0.5_\wedge\overline{)0.6_\wedge31}$

(d) Move both decimal points 3 places to the right.

$0.031_\wedge\overline{)92.000_\wedge}$

THREE ZEROS MUST BE ADDED

(e) Move both decimal points 2 places to the right.

```
3.15^ |0.76^3
```

(f) Move both decimal points 1 place to the right.

```
42.6^ |50(0)^
```

ONE ZERO MUST BE ADDED

2. (a) First, move the decimal point 3 places to the right in both numbers and locate the decimal in the answer.

```
             375.
0.136^ |51.000^
        40 8
        10 20
         9 52
           680
           680
             0
```

DECIMAL POINT AND CARET LINE UP

(b) First, move the decimal point 1 place to the right in both numbers and locate the decimal in the answer.

```
         2 0.4
34.1^ |695.6^4
       682
        1 3 6
            0
        1 3 6 4
        1 3 6 4
              0
```

DECIMAL POINT AND CARET LINE UP.

Exercise #15, page 40

1.

```
             .015
6.1^ |0.0^950
           61
           340
           305
            35
```

NOTE THAT THIS REMAINDER IS MORE THAN HALF OF 61.

Rounded quotient is 0.016

LAST DIGIT IN QUOTIENT HAS BEEN INCREASED BY 1.

2.

```
          6 09.6
0.52^ |317.00^0
       312
         5 0
           0
         5 0 0
         4 6 8
           3 2 0
           3 1 2
               8
```

NOTE THAT THIS REMAINDER IS LESS THAN HALF OF 52.

Rounded quotient is 609.6

LAST DIGIT IN QUOTIENT HAS BEEN LEFT UNCHANGED.

Exercise #16, page 42

1. (a) The decimal point must be moved 1 place to the left (since 10 has 1 zero).

   (b) The decimal point must be moved 4 places to the left (since 10 000 has 4 zeros).

   (c) The decimal point must be moved 2 places to the left (since 100 has 2 zeros).

   (d) The decimal point must be moved 3 places to the left (since 1000 has 3 zeros).

   (e) The decimal point must be moved 6 places to the left (since 1 000 000 has 6 zeros).

2.

Number	Divide by		
	10	100	1000
(a) 62.5	6.25	0.625	0.0625
(b) 837.46	83.746	8.374 6	0.837 46
(c) 6.9	0.69	0.069	0.006 9
(d) 16 439	1643.9	164.39	16.439
(e) 0.01	0.001	0.000 1	0.000 01

3. (a) 82.74 ÷ 100 = 0.827 4 (Move decimal point in 82.74 two places to the left.)

   (b) 33.52 ÷ 1000 = 0.033 52 (Move decimal point in 33.52 three places to the left.)

   (b) 0.05 ÷ 10 = 0.005 (Move decimal point in 0.05 one place to the left.)

   (d) 6843.2 ÷ 10 000 = 0.684 32 (Move decimal point in 6843.2 four places to the left.)

   (e) 23.4 × 100 = 2340 (Move decimal point in 23.4 two places to the right.)

   (f) 6.9 × 1000 = 6900 (Move decimal point in 6.9 three places to the right.)

   (g) 0.07 ÷ 1000 = 0.000 07 (Move decimal point in 0.07 three places to the left.)

   (h) 2.115 × 10 = 21.15 (Move decimal point in 2.115 one place to the right.)

End of Lesson 5

Lesson 6

Real Numbers

Basic Algebra and Geometry

REAL NUMBERS

Topic One: Relationship Between Decimal Numbers and Rational Numbers

A. Writing Rational Numbers as Decimals

In Lesson 4, you learned that the rational number $\frac{a}{b}$ represents the quotient $a \div b$. We can use this fact to help us convert a rational number to a decimal number. Any rational number can be written in decimal form by dividing the denominator into the numerator.

EXAMPLE: The rational number $\frac{3}{8}$ can be written in decimal form by dividing 8 into 3. (Note that zeros are inserted to the right of the decimal point in the dividend, 3.)

```
       .375
 8  )3.000 <——Keep bringing down zeros until you get a
     2 4         remainder of 0.
      60
      56
       40
       40
        0 <——Division is complete since a remainder of
                 0 is obtained.
```

In this case, the division process comes to an end and a decimal with a finite number of digits is obtained. Decimals of this type are called TERMINATING DECIMALS. The rational number $\frac{3}{8}$ can be represented by the terminating decimal 0.375.

The following rational numbers are all equivalent to terminating decimals. In each case, divide the denominator into the numerator and continue the division until you obtain a remainder of zero. Then, state the terminating decimal that is equivalent to each rational number.

NEGATIVE NUMBERS

1. $\frac{^-106}{25} = \underset{\text{(decimal)}}{\underline{^-4.24}}$

```
       4.24
25 | 106.00
     100
       60
       50
       100
       100
         0
```

2. $\frac{7}{20} = \underset{\text{(decimal)}}{\underline{\qquad\qquad}}$

```
20 | 7.
```

3. $\frac{-5}{16} = \underline{\qquad\qquad}$ (decimal)

4. $\frac{43}{4} = \underline{\qquad\qquad}$ (decimal)

$16 \overline{)5.0000}$

$4 \overline{)43.}$

In some cases when you divide the denominator of a rational number into the numerator, you will be unable to obtain a remainder of zero. The division could be carried on forever, but you would never get a zero remainder.

EXAMPLE: The rational number $\frac{2}{11}$ can be written in decimal form by dividing 11 into 2. (Note that the digits "18" start repeating in the dividend.)

```
       .1818...  ← THIS SEQUENCE OF DIGITS REPEATS.
 11   )2.0000
       1 1
         90
         88
          20
          11
           90 ← At this point, the remainders start
           88   repeating. (i.e. this is the second
            2   time you have had to divide 11 into
                90.)
```

In this case, the division process is never complete but the remainders start repeating. You obtain a decimal with an infinite number of digits but these digits follow a recurring pattern. Decimals of this type are called NON-TERMINATING REPEATING DECIMALS. The rational number $\frac{2}{11}$ can be represented by the non-terminating repeating decimal $0.\overline{18}$. Note that a bar is placed above the two digits 1 and 8 to indicate that this sequence repeats.

The following rational numbers are all equivalent to non-terminating repeating decimals. In each case, divide the denominator into the numerator and continue the division until you notice a certain sequence of digits that recurs in the quotient. Then, state the repeating decimal that is equivalent to each rational number.

1. $\frac{-23}{6} = \underset{\text{(decimal)}}{-3.8\overline{3}}$ PUT A BAR ABOVE THE DIGIT THAT REPEATS.

$$\begin{array}{r} 3.833\ldots \\ 6\overline{)23.000} \\ \underline{18} \\ 5\;0 \\ \underline{4\;8} \\ 20 \\ \underline{18} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

THIS DIGIT KEEPS REPEATING.

2. $\frac{5}{18} = \underset{\text{(decimal)}}{\underline{\qquad\qquad}}$

$18\overline{)5.000}$

3. $\frac{-31}{15} = \underset{\text{(decimal)}}{\underline{\qquad\qquad}}$

$15\overline{)31.}$

4. $\frac{5}{11} = \underset{\text{(decimal)}}{\underline{\qquad\qquad}}$

$11\overline{)5.}$

These examples illustrate the following points.

1. When the denominator of a rational number is divided into the numerator, a remainder occurs at each step. If at one stage of the division the remainder is zero, the division is complete and a terminating decimal is obtained.

2. If a remainder of zero is not obtained, some remainder will occur for a second time and a non-terminating repeating decimal is obtained.

In general, every rational number can be expressed as a terminating or a non-terminating repeating decimal.

B. Writing Decimals as Rational Numbers

Now, let us try to reverse the procedure and write decimals as rational numbers.

Any terminating decimal may be written as a rational number of the form $\frac{a}{b}$ where b is 10 or a power of 10. Review pages 21 and 22 of Lesson 5 where you were shown how to do this.

EXAMPLES:

$0.37 = \frac{37}{100}$ ← THESE ARE THE DIGITS IN THE DECIMAL. ← THE LAST DIGIT 7 IS IN THE HUNDREDTHS PLACE.

$-3.009 = -3\frac{9}{1000}$ ← .009 IS EQUIVALENT TO $\frac{9}{1000}$.

THESE NUMBERS CORRESPOND.

$= \frac{-3009}{1000}$

Give an equivalent rational number for each of the following terminating decimals.

1. $-0.0309 = \frac{-309}{10,000}$
2. $0.9 =$ ______
3. $0.73 =$ ______
4. $-0.017 =$ ______
5. $0.07 =$ ______
6. $-0.0023 =$ ______

Write each decimal as a mixed number and then as a rational number.

	Decimal	Mixed Number	Rational Number
1.	-18.13	$-18\frac{13}{100}$	$\frac{-1813}{100}$
2.	10.7	______	______
3.	-1.03	______	______
4.	1.901	______	______
5.	66.67	______	______

Any non-terminating repeating decimal may also be written as a rational number. Study the following examples.

EXAMPLE 1: Change $1.\overline{3}$ to a rational number.
Solution

Let x represent the decimal $1.\overline{3}$.

$$x = 1.\overline{3} = 1.3333...$$

MULTIPLY BY 10 SINCE ONLY ONE DIGIT REPEATS IN THE DECIMAL $1.\overline{3}$.

$$\begin{aligned} 10x &= 13.3333... \\ x &= 1.3333... \\ \hline 9x &= 12 \quad \text{(subtract)} \end{aligned}$$

$$x = \frac{12}{9} \text{ or } \frac{\quad}{3}$$ FILL IN.

The non-terminating repeating decimal $1.\overline{3}$ can be represented by the rational number $\frac{\quad}{3}$.

EXAMPLE 2: Change $0.\overline{732}$ to a rational number.
Solution

Let x represent the decimal $0.\overline{732}$.

$$x = 0.\overline{732} = 0.732\ 732...$$

MULTIPLY BY A 1000 SINCE A SEQUENCE OF THREE DIGITS REPEATS IN $0.\overline{732}$.

$$\begin{aligned} 1000x &= 732.732\ 732... \\ x &= 0.732\ 732... \\ \hline 999x &= 732 \quad \text{(subtract)} \end{aligned}$$

$$x = \frac{732}{999} \text{ or } \frac{\quad}{333}$$

The non-terminating repeating decimal $0.\overline{732}$ can be represented by the rational number $\frac{\quad}{333}$.

Give an equivalent rational number for each of the following non-terminating repeating decimals.

1. $-4.\overline{23}$

Let x represent the decimal $4.\overline{23}$.

$$x = 4.\overline{23} = 4.2323\ldots$$

$$\begin{array}{rl} 100x = & \\ x = & \\ \hline 99x = & \underline{\qquad} \text{ (subtract)} \end{array}$$

$$x = \frac{\qquad}{99}$$

$$-4.\overline{23} = \frac{\qquad\qquad}{\text{(rational no.)}}$$

2. $0.11\overline{2}$

Let x represent the decimal $0.11\overline{2}$.

$$x = 0.11\overline{2} = 0.112\ 222\ldots$$

$$\begin{array}{rl} 10x = & 1.122\ 22\ldots \\ x = & \\ \hline \underline{\quad}x = & 1.01 \qquad \text{(subtract)} \end{array}$$

$$x = \frac{1.01}{\square}$$

Multiply both terms by 100.

$$x = \frac{101}{\square}$$

$$0.11\overline{2} = \frac{\qquad\qquad}{\text{(rational no.)}}$$

Thus, we have found that both terminating and repeating decimals can be written in rational form.

In general, every terminating or non-terminating repeating decimal can be expressed as a rational number.

C. <u>One-to-One Correspondence Between Rational Numbers and Decimals</u>

We can take the union of the set of terminating decimals and the set of non-terminating repeating decimals and form the set of PERIODIC DECIMALS. There is a one-to-one correspondence between the set of periodic decimals and the set of rational numbers. This means that:

1. Every rational number can be associated with exactly one periodic decimal.

2. Every periodic decimal can be associated with exactly one rational number.

Thus, all the properties that apply to the set of rational numbers must also apply to the set of periodic decimals. Review the chart on page 48, lesson 4, and name the property which justifies each of the following statements.

Statement	Justification
1. $(0.5 + 0.3)$ is a rational number.	closure prop. of addition
2. $0.\overline{3} + 0.\overline{6} = 0.\overline{6} + 0.\overline{3}$	______
3. $0 + 3.5 = 3.5$	______
4. $1.83 \times \frac{1}{1.83} = 1$	______
5. $6 \times (3.8 \times 9.73) = (6 \times 3.8) \times 9.73$	______
6. $13.2 + (-13.2) = 0$	______

EXERCISE - Relationship Between Decimal Numbers and Rational Numbers

1. Fill in the blanks.

(a) The union of the set of terminating decimals and the set of non-terminating repeating decimals is the set of ______ decimals.

(b) A rational number can be changed to a decimal by dividing the ______ of the fraction into the ______.

(c) Decimals that have a finite number of digits are called ______ decimals.

(d) There is a one-to-one correspondence between the set of periodic decimals and the set of ______ numbers.

(e) A decimal that contains an infinite number of digits, some of which follow a recurring pattern, is called a non-______ ______ decimal.

(f) The decimal 0.37 can be written as the rational number ______.

(g) The rational number $\frac{7}{3}$ can be written as the decimal number ______.

2. Express the following rational numbers as periodic decimals. State whether the decimals are terminating or non-terminating repeating. Space has been provided below for your calculations.

(a) $\frac{131}{3}$ = $43.\overline{6}$ which is a non-terminating repeating decimal.

(b) $\frac{13}{8}$ = ______ which is a ______________________ decimal.

(c) $\frac{1}{12}$ = ______ which is a ______________________ decimal.

(d) $\frac{11}{15}$ = ______ which is a ______________________ decimal.

(e) $\frac{9}{40}$ = ______ which is a ______________________ decimal.

Calculations

(a)	(b)	(c)	(d)	(e)
43.66... 3)131.00 12 11 9 20 18 20 18 2				

3. Express the following periodic decimals as rational numbers in lowest terms.

(a) $0.\overline{45}$

Let x represent the decimal $0.\overline{45}$.

$x = 0.\overline{45} = 0.4545...$

$100x =$

$x =$

(b) 0.164

(c) $0.\overline{123}$ (d) 0.0025

Topic Two: From Rational Numbers to Real Numbers

A. Irrational Numbers

There is a one-to-one correspondence between the set of rational numbers and the set of periodic decimals. (Remember that a periodic decimal either terminates or repeats.) It is easy, however, to make up decimals which are not periodic in form.

0.505 005 000 500 005...

The above decimal is formed by writing a 5 after the decimal point and following this by a zero, then writing 5 followed by two zeros, 5 followed by three zeros, and so on. This decimal is NON-PERIODIC since it neither terminates nor repeats. The decimals listed below are all non-periodic decimals.

$$\left.\begin{array}{l} 0.635\ 729\ 984\ 37\ldots \\ 5.494\ 994\ 999\ 4\ldots \\ 0.123\ 456\ 789\ 001\ 122\ 3\ldots \\ 3.678\ 439\ 716\ 25\ldots \end{array}\right\}$$

These are infinite decimals that do not have a recurring sequence of digits.

Give an example of another decimal that is non-periodic.

____

The set of numbers which can be represented by non-periodic decimals is called the set of IRRATIONAL NUMBERS. (The prefix "ir" means "not".) For convenience, we will always designate this set by the capital letter $\overline{Q}$. Do not make the assumption that there is a certain "inexactness" associated with an irrational number. An irrational number is just as precise or exact as a rational number.

Classify each of the following decimals.

		Periodic or Non-Periodic?	Rational or Irrational?
1.	$5.\overline{445}$	periodic	rational
2.	126.738 52...	non-periodic	irrational
3.	0.3333...		
4.	0.101 001 000 1...		
5.	12.036 5		
6.	0.037 999 99...		
7.	6.123 847 26		
8.	14.035 353 5...		
9.	0.325 325 532 555 3...		
10.	0.51		

B. Square Roots

The operation of finding a square root involves finding a number which when multiplied by itself equals a given number. For example, finding the square root of 49 involves finding a number which when multiplied by itself equals 49.

Note that:

$$7 \times 7 = 49$$
$$\text{and } -7 \times -7 = 49$$

Thus, 7 and -7 are both square roots of 49. These two square roots of 49 are equal in absolute value but opposite in sign.

(a) $\sqrt{49}$ represents the PRINCIPAL SQUARE ROOT of 49 which happens to be 7.

(b) $-\sqrt{49}$ represents the NEGATIVE SQUARE ROOT of 49 which happens to be -7 .

In general, <u>any</u> positive number has <u>two</u> square roots that are equal in absolute value but opposite in sign.

EXAMPLES:

1. $\sqrt{5}$ represents the principal square root of 5.
2. $-\sqrt{9}$ represents the negative square root of 9.
3. The square root of 8 is $\sqrt{8}$ or $-\sqrt{8}$. The principal square root is $\sqrt{8}$ and the negative square root is $-\sqrt{8}$.

Fill in the blanks below.

1. $\sqrt{16}$ represents the ______________ square root of ______.
   $\sqrt{16}$ is equal to the integer ___ since 4 × ___ = 16.
2. $-\sqrt{81}$ represents the ______________ square root of ______.
   $-\sqrt{81}$ is equal to the integer ___ since -9 × ___ = 81.
3. The square root of 27 is ___ or $-\sqrt{27}$. The principal square root is _____ and the negative square root is _____.
4. The square root of 25 is equal to the integers 5 and ___. The principal square root is the integer _____ and the negative square root is the integer _____.

The square root of a positive number may be either a rational number or an irrational number. Any number that has a rational square root is called a PERFECT SQUARE. For example, 64, 121, 2500, 0.36, and $\frac{4}{9}$ are all perfect squares since they have rational square roots.

i.e.

$\sqrt{64} = 8$ and $-\sqrt{64} = -8$

$\sqrt{121} =$ ___ and $-\sqrt{121} =$ ___

$\sqrt{2500} =$ ___ and $-\sqrt{2500} = -50$

$\sqrt{0.36} = 0.6$ and $-\sqrt{0.36} =$ ___

$\sqrt{\frac{4}{9}} = \frac{2}{3}$ and $-\sqrt{\frac{4}{9}} = \frac{-2}{3}$

Fill in the blanks.

All these square roots are rational numbers.

Name five other real numbers that are perfect squares.

8100, ______, ______, ______, ______

Fill in each blank with the rational number that the square root represents. Check each answer by multiplying it by itself to see if you obtain the number under the square root sign.

	Check
1. $-\sqrt{0.0004}$ = -0.02	$-0.02 \times -0.02 = 0.0004$
2. $\sqrt{144}$ = ____	____
3. $\sqrt{0.49}$ = ____	____
4. $\sqrt{0.0016}$ = ____	____
5. $-\sqrt{\frac{9}{16}}$ = ____	____
6. $\sqrt{10\ 000}$ = ____	____

Although some square roots are rational numbers, others are irrational. For example, the symbol $\sqrt{2}$ represents the principal square root of 2 and is an irrational number. $\sqrt{2}$ represents the same irrational number as the non-periodic decimal 1.414 213 5... . Mathematicians have worked out $\sqrt{2}$ to a great number of decimal places, but this decimal will neither terminate nor repeat; hence it is non-periodic.

The square roots listed below are all irrational numbers that are equal to non-periodic decimals.

$$\left.\begin{aligned} -\sqrt{15} &= -3.8729\ldots \\ \sqrt{201} &= 14.1774\ldots \\ \sqrt{32} &= 5.6567\ldots \end{aligned}\right\}$$ The digits in these decimals go on and on but have no repetend.

Name five other square roots that are irrational numbers.

$\sqrt{65}$, ______, ______, ______, ______.

SQUARE ROOT OF A POSITIVE NUMBER

Every positive number "a" has two square roots that are either both rational or both irrational.

$\sqrt{a}$ represents the principal square root of a.
$-\sqrt{a}$ represents the negative square root of a.

Can we find a rational or irrational number that is the square root of a negative number? For example, what is the square root of -49? In determining the square root of -49, we must find a number which when multiplied by itself equals -49. That is, we must find a number n so that

$$n \times n = -49$$

We cannot find a member of set Q or set $\overline{Q}$ that satisfies this condition. We know that the product of two positive numbers or two negative numbers is always a positive number. Thus, the product of two identical factors cannot yield a negative product. The expressions $\sqrt{-49}$ and $-\sqrt{-49}$ are not rational or irrational numbers. (i.e. they do not belong to either set Q or set $\overline{Q}$.)

SQUARE ROOT OF A NEGATIVE NUMBER
For any negative number "a", $\sqrt{a}$ and $-\sqrt{a}$ do not represent rational or irrational numbers.

Self-correcting Exercise #1

Answers may be found on page 51 of this lesson.

1. State whether each expression represents a rational number, irrational number, or neither of these.

(a) $\sqrt{625}$ ________	(b) $\sqrt{0.1}$ irrational	(c) $-\sqrt{0.01}$ ________
(d) $\sqrt{19}$ ________	(e) $\sqrt{-19}$ ________	(f) $-\sqrt{81}$ ________
(g) $-\sqrt{-81}$ ________	(h) $\sqrt{0}$ ________	(i) $-\sqrt{1}$ ________
(j) $\sqrt{-1}$ ________	(k) $\sqrt{\frac{3}{4}}$ ________	(l) $\sqrt{\frac{9}{4}}$ ________

2. Write each root using a radical sign and then simplify the root if possible.

   (a) Square root of 81.

   (b) Square root of 7.

   (c) Principal square root of $\frac{1}{4}$.

(d) Negative square root of 64.

(e) Principal square root of 19.

C. The Irrational Number π

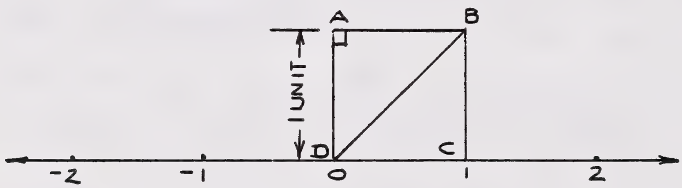
Mathematicians have found that the ratio between the circumference of a circle and its diameter is a constant value that is a little more than 3. This quotient is the same for all circles and is equivalent to the non-periodic decimal 3.141 592 6... . Since this decimal is non-repeating and infinite it cannot be written in its entirety and is represented by the Greek letter π (pronounced "pi"). π is an irrational number.

D. Graphing Irrational Numbers on the Number Line

Irrational numbers are coordinates of definite points on the number line.

The irrational numbers $\sqrt{2}$ and $-\sqrt{2}$ can be located on the number line by using the following procedure.

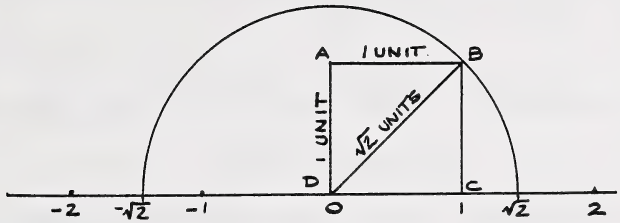
1. Draw square ABCD, of sides 1 unit, on the number line so that point D falls at the origin and point C falls on 1. Join points B and D to form a diagonal of the square.



2. Triangle ABD is a right triangle. The Pythagorean Theorem states that the square on the longest side of a right triangle is equal to the sum of the squares on the other two sides. We can use this theorem to find the length of side BD of right triangle ABD.

$$\begin{aligned}(BD)^2 &= (AD)^2 + (AB)^2\\ (BD)^2 &= (\underline{\quad})^2 + (1)^2\\ (BD)^2 &= 2\\ BD &= \sqrt{2}\end{aligned}$$

3. With your compass centred at the origin and using a radius the length of diagonal BD ($\sqrt{2}$), describe a semi-circle that cuts the number line in two places. The point of intersection to the right of the origin corresponds to the irrational number $\sqrt{2}$ and the point of intersection to the left of the origin corresponds to the irrational number _____.



What two integers does $\sqrt{2}$ lie between? ____________

Which of these integers is it closer to? ______

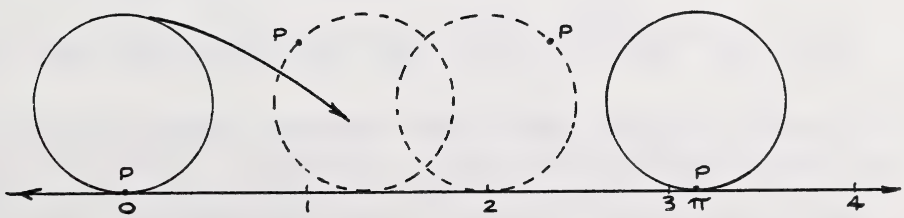
What two integers does $-\sqrt{2}$ lie between? ____________

Which of these integers is it closer to? ______

The irrational number π can also be located on the number line. Consider a circle with a diameter of 1 unit. Its circumference is

$$\begin{aligned} C &= \pi d \\ &= \pi \times 1 \\ &= \pi \text{ units} \end{aligned}$$

If such a circle is rolled along a number line for one complete revolution starting with point P on the circumference coinciding with the origin on the number line, a point corresponding to π can be determined.



The point corresponding to π lies between what two integers on the number line? ______ Which of these integers is it closer to? ______

Theoretically, we can locate any irrational number on the number line by considering rational approximations to it. For example, we could use the following procedure to locate the irrational number

2.732 685 4...

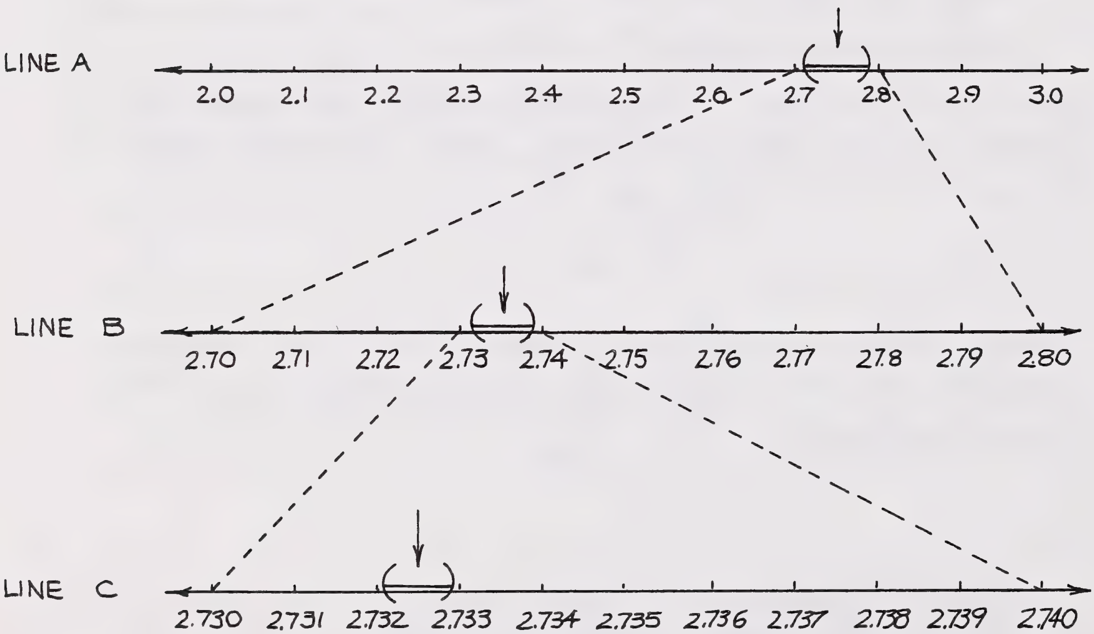
1. Consider the first three digits in the number. (2.73) They tell us that the number is located between the two rational numbers 2.7 and 2.8. (See line A below.)

2. Consider the first four digits in the number. (2.732) They tell us that the number is located between the two rational numbers

   2.73 and __________ (See line B below.)

3. Consider the first five digits in the number. (2.7326) They tell us that the number is located between the two rational numbers

   2.732 and ________ (See line C below.)



4. By looking at more digits in the irrational number 2.732 685 4..., we can locate this number more accurately. If this process were carried on indefinitely, we could find exactly one point which corresponds to the irrational number 2.732 685 4... .

In the exercises below, decide if each statement is true or false.

1. The point corresponding to the irrational number 6.943 872... lies between:

	True or False?
(a) 7 and 8	false
(b) 6.8 and 6.9	
(c) 6.94 and 6.95	
(d) 6.943 and 6.944	
(e) 6.93 and 6.94	
(f) 6.9 and 7.0	
(g) 6.9438 and 6.9439	

2. The point corresponding to the irrational number 0.062 187... lies between:

	True or False?
(a) 0.06 and 0.07	true
(b) 0 and 1	
(c) 0.1 and 0.2	
(d) 0.01 and 0.02	
(e) 0.062 and 0.063	
(f) 0.6 and 0.7	
(g) 0.062 18 and 0.062 19	

> Every irrational number can be represented by a point on the number line.

E. Rational Approximations to Irrational Numbers

It must be understood that an irrational number is an exact number. For example, the irrational number $\sqrt{2}$ is an exact number and often appears in mathematics in this form. However, when an irrational number is required in decimal form for computation purposes, the value is approximated by using a rational number. For example, the value of the irrational number $\sqrt{2}$ is often approximated by the rational number 1.414. The number 1.414 is said to be a RATIONAL APPROXIMATION of $\sqrt{2}$. Similarly, the rational approximations

$\frac{22}{7}$, 3.14, 3.142, 3.1416 are often used for the irrational number π. We use the symbol "$\doteq$" between two numbers to indicate that they are approximately equal to each other.

$$\pi \doteq 3.14$$
$$\sqrt{2} \doteq 1.414$$

On page 50 of this lesson, you will find a table of square roots for integers between 1 and 100 inclusive. In this table, some values of $\sqrt{n}$ are exact while others are only rational approximations. For example, $\sqrt{25}$ is exactly equal to 5. (i.e. $\sqrt{25}$ is a rational number that is equal to 5.) On the other hand, 5.196 is only a rational approximation of $\sqrt{27}$ (i.e. $\sqrt{27}$ is an irrational number that is approximately equal to 5.196.)

Use the table of square roots to write the following numbers in decimal form. Then decide if each decimal is exactly equal to the square root or only approximately equal to it.

Square Root	Decimal Value	Exact or Approximate?
$-\sqrt{50}$	-7.071	approximate
$\sqrt{49}$	______	______
$\sqrt{100}$	______	______
$-\sqrt{68}$	______	______
$\sqrt{24}$	______	______
$\sqrt{1}$	______	______
$-\sqrt{72}$	______	______

The table of square roots can be used to help us compute with irrational numbers.

EXAMPLE: Using the table of square roots, simplify the expression $5(\sqrt{3}+\sqrt{2})$.

Solution

SYMBOL FOR "IS APPROXIMATELY EQUAL TO"

According to the table, $\sqrt{3} \doteq 1.732$ and $\sqrt{2} \doteq 1.414$

Thus,

$$\begin{aligned} 5(\sqrt{3}+\sqrt{2}) &\doteq 5(1.732 + 1.414) \\ &\doteq 5(3.146) \\ &\doteq \underline{15.730} \end{aligned}$$

Using the table of square roots on page 50 of this lesson, represent each expression by a single decimal number.

1. $13 + \sqrt{7}$

$\doteq$ __ + _____

$\doteq$ __________

2. $3\sqrt{18}$

$\doteq$ __ × _____

$\doteq$ __________

3. $3\sqrt{6} + 5\sqrt{8}$

$\doteq$ (3 × ____) + (5 × ____)

$\doteq$ _______ + _______

$\doteq$ __________

4. $\dfrac{\sqrt{94} - \sqrt{89}}{3}$

$\doteq \dfrac{(\quad\quad) - (\quad\quad)}{3}$

$\doteq \dfrac{\_\_\_\_\_\_}{3}$

$\doteq$ __________

F. <u>The Set of Real Numbers</u>

The set of rational numbers which can be represented by periodic decimals and the set of irrational numbers which can be represented by non-periodic decimals are DISJOINT sets (i.e. they do not have any elements in common).

$$\{\text{all rational numbers}\} \cap \{\text{all irrational numbers}\} = \emptyset$$
$$Q \cap \bar{Q} = \emptyset$$

If we form the union of the set of rational numbers and the set of irrational numbers, we will arrive at a new set of elements which we can call the set of REAL NUMBERS. For convenience, we will always designate this set by the capital letter R.

$$\{\text{all real numbers}\} = \{\text{all rational numbers}\} \cup \{\text{all irrational numbers}\}$$
$$R = Q \cup \bar{Q}$$

The following diagram illustrates the composition of set R.

Real Number (R)

Rational Numbers (Q) Irrational Numbers ($\overline{Q}$)

Although the set of real numbers seems to encompass all the numbers we have dealt with so far, remember that square roots of negative numbers are <u>not</u> real numbers. For example, $\sqrt{-1}$, $\sqrt{-36}$, $\sqrt{-0.4}$, $\sqrt{\frac{-25}{4}}$, etc. do not belong to set R.

Self-correcting Exercise #2

Answers may be found on page 51 of this lesson.

1. Classify each number as being "real" or "not real". Then, for each real number, state whether it is "rational" or "irrational".

(a) 3.7 real (rational)

(b) 1.235 728 643...

(c) $\sqrt{-17}$

(d) 0

(e) $\sqrt{0.16}$

(f) $\frac{-7}{8}$

(g) π

(h) 4.3207

(i) $\sqrt{\frac{-9}{16}}$

(j) $\sqrt{0.465}$

(k) $5\frac{7}{8}$

G. The Real Number Line

In Lesson 4, we assigned rational numbers to points on a number line. Because there are an infinite number of rational numbers, one is tempted to assume that the rational numbers fill up the entire number line. But, the rational numbers are dense, but not complete. This means that between any two rational numbers another rational number can always be located, but the rational numbers do not fill up all the points on a solid number line. If the entire set of rational numbers were located on the number line, there would still be some points in the line which had no rational numbers assigned to them. These points represent irrational numbers like $\sqrt{2}$, π, 2.732 684..., and so on.

A line that has points associated with real numbers is called a REAL NUMBER LINE. Such a line includes points which represent rational numbers as well as those which represent irrational numbers.

There is a one-to-one correspondence between the set of real numbers and the set of points in a solid number line. This means that:

1. Every real number can be represented by a point in a solid number line.

2. No matter where you may locate a point in the number line, there is a real number that can be associated with that point.

Thus, we can say that the real numbers are COMPLETE.

COMPLETENESS PROPERTY
There is a one-to-one correspondence between the set of real numbers and the set of points in a line.

Like the rational numbers, the real numbers have the density property.

DENSITY PROPERTY
Between any two real numbers, there is another real number.

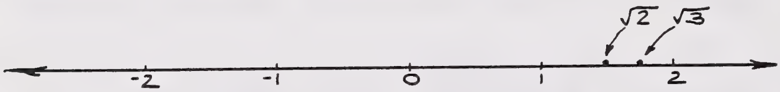
The real numbers are ordered in a similar manner to the way natural, whole, integral, and rational numbers are ordered. For any two real numbers a and b, if point a lies to the right of point b, then a is greater than b ($a > b$). If point a lies to the left of point b, then a is less than b ($a < b$). If points a and b coincide, then a and b are equal ($a = b$).

EXAMPLE: Of the two numbers $\sqrt{2}$ and $\sqrt{3}$, which is larger?

Solution

We must look at rational approximations of the irrational numbers $\sqrt{2}$ and $\sqrt{3}$ to determine how they are ordered.

$$\sqrt{2} \doteq 1.414$$
$$\sqrt{3} \doteq 1.732$$



Since $1.732 > 1.414$, $\sqrt{3} > \sqrt{2}$. (Note that $\sqrt{3}$ lies to the right of $\sqrt{2}$ on the number line.)

Self-correcting Exercise #3

Answers may be found on page 52 of this lesson.

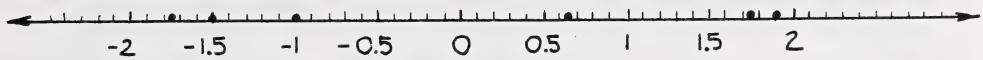
1. Give decimal values for each of the following real numbers. Then choose the number in each pair that is larger. (Use table at the end of this lesson to evaluate the square roots.)

	Real Numbers		Decimal Values	Larger Number
(a)	$1\frac{1}{2}$,	$\sqrt{2}$	1.5 , ______	$1\frac{1}{2}$
(b)	$-\sqrt{5}$,	$-\sqrt{6}$	______ , ______	______
(c)	π;	$\sqrt{10}$	______ , ______	______
(d)	$\frac{15}{11}$,	$\sqrt{3}$	______ , ______	______
(e)	$\sqrt{62}$,	$\frac{63}{8}$	______ , ______	______

H. Graphing Subsets of R

The graph of a given subset of the set of real numbers is the set of points in the real number line whose coordinates are members of the subset.

If the subset contains a finite number of elements, the graph will consist of a number of discrete points. For example, the graph of $\{-1, -1\frac{1}{2}, 1.9, 0.\bar{6}, -\sqrt{3}, 1.76\}$ consists of six points on the real number line.



If the subset you are graphing contains a continuous interval of real numbers, the graph will consist of an entire section of the number line.

EXAMPLE 1: Graph the set of all real numbers that are greater than or equal to -3.

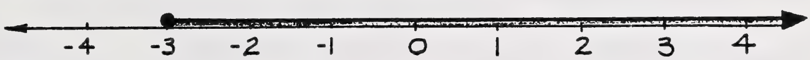
Solution

In set builder notation, this set could be specified as follows:

$$\{x \mid x \geqq -3, \ x \in R\}$$

This is read, "the set of all x such that x is greater than or equal to -3 and x is a real number."

When graphing this set, we place a solid dot at -3 to indicate that the point corresponding to -3 is included in the graph. The graph of this set must also contain all the points that correspond to real numbers that are greater than -3. Since the real numbers are continuous, we can draw a solid line to the right of -3 to represent all the real numbers greater than -3. We must place an arrow at the end of this line to indicate that the graph continues indefinitely to the right.



Is $\sqrt{2}$ included in the set that is graphed above? _____ ($\sqrt{2} \doteq 1.414$)

Is $-\sqrt{2}$? _______ Is $-\sqrt{9}$? _______ Is $-\sqrt{10}$? _______

Is 4.65? _______ Is $\sqrt{87}$? _______ Is 0? _______

Is -0.3? _______ Is $\frac{18}{5}$? _______ Is $-\pi$? _______

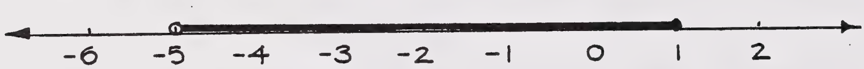
EXAMPLE 2: Graph the set of real numbers that are greater than -5 and less than or equal to 1.

Solution

In set-builder notation, this set could be specified as follows:

$$\{x \mid -5 < x \leqq 1, x \in R\}$$

When graphing this set, we place a hollow dot at -5 to indicate that the point corresponding to -5 is not included in the graph. We place a solid dot at 1 to indicate that the point corresponding to 1 is included in the graph. Since the real numbers are complete, we can draw a solid line between the dots at -5 and 1 to represent all the real numbers that are between -5 and 1.



Is $-\sqrt{22}$ included in the set that is graphed above? yes ($-\sqrt{22} \doteq -4.690$)

Is $\frac{4}{3}$? ______ Is $-\sqrt{6}$? ______ Is $-\pi$? ______ Is π? ______

Is $\sqrt{2}$? ______ Is $-\sqrt{25}$? ______ Is $\sqrt{1}$? ______

Is $\frac{-39}{8}$? ______ Is 0.3? ______

Self-correcting Exercise #4

Answers may be found on page 52 of this lesson.

1. Use set-builder notation to specify each set, and then graph it on the number line provided.

(a) The set of all real numbers between -1 and 4.

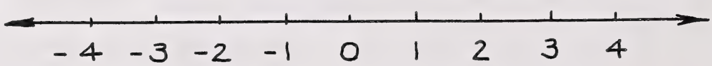
Set-Builder Notation: ____________________

Graph:

(b) The set of all real numbers that are less than -1.

Set-Builder Notation: ____________________

Graph:



(c) The set of all real numbers that are less than or equal to -2 <u>or</u> greater than or equal to 2. (Hint: This set consists of <u>two</u> continuous intervals.)

Set-Builder Notation: ____________________

Graph: 

I. The Real Number System

The set of real numbers together with the operations of addition and multiplication and some of their properties form the REAL NUMBER SYSTEM. The real numbers satisfy all the properties we have already attributed to the rational numbers under the operations of addition and multiplication.

Fill in the blanks in the chart below.

For any three real numbers a, b, and c ($a, b, c \in R$):

	Addition	Multiplication
Closure properties	a + b is a real number.	________ is a ________ number.
________ properties	a + b = b + a	ab = ____
________ properties	a + (b + c) = (__ + __) + c	________ = ________
________ property	a(b + c) = ___ + ___	
Identity elements	Additive identity is ___. a + ___ = a	Multiplicative identity is ___. a × ___ = a
Inverse elements	Additive inverse of "a" is ___. a + ___ = 0	Mult. inverse of "a" is ___. a × ___ = 1

Since the real number system satisfies the same properties as the rational number system, it also forms a FIELD.

J. Relationship Between Sets N, W, I, Q, and R

In this course we have talked about many different kinds of numbers. Let us review some of the more important sets of numbers and discuss how they are related to each other.

1. N = {1, 2, 3, . . .} is the set of natural numbers.

   Does 0 belong to set N? _____ Does 5? _____ Does -2? _____

   Does 1 000? _____ Does $3\frac{1}{2}$? _____ Does $\sqrt{3}$? _____

2. W = {0, 1, 2, 3, . . .} is the set of whole numbers.

   This set differs from the natural numbers in only one respect; it contains the element __________.

3. I = {. . ., -3, -2, -1, 0, 1, 2, 3, . . .} is the set of integers.

   Does 0 belong to set I? _____ Does 5? _____ Does -5? _____

   Does $2\frac{1}{2}$? _____ Does $\sqrt{2}$? _____ Does -4.2? _____

   The set of natural numbers and the set of whole numbers are both subsets of the set of integers. This means that every natural number and every whole number is also an integer. For example, 7 is a natural number, whole number, and integer. But, set I includes negative integers that do not belong to set N or set W. For example, -7 is an integer, but it is not a natural number or a whole number.

   Is 2 a natural number? _____ a whole number? _____ an integer? _____

   Is -2 a natural number? _____ a whole number? _____ an integer? _____

   Is 0 a natural number? _____ a whole number? _____ an integer? _____

4. Any number which can be expressed in the form $\frac{a}{b}$, where a and b are integers ($b \neq 0$) belongs to Q, the set of rational numbers. For example, $\frac{2}{3}$, $\frac{1}{5}$, $\frac{-1}{2}$, $\frac{-6}{7}$ are all rational numbers. You will recall that any periodic decimal (i.e. any decimal that terminates or repeats) may be written in rational

form. Thus, decimals like 6.5, $0.0\overline{4}$, 0.0056, $12.\overline{317}$ are all rational numbers. On the other hand, non-periodic decimals like 0.050 050 005... and 2.638 579... are not rational numbers.

Does $\frac{1}{4}$ belong to set Q? _____ Does 3.98? _____ Does $\sqrt{2}$? _____

Does π? _____ Does $\frac{39}{7}$? _____ Does $106.\overline{8}$? _____

Does 1.436 78...? _____ Does $\frac{-18}{3}$? _____ Does $2\frac{3}{4}$? _____

The set of integers is a subset of the set of rational numbers. This means that every integer is also a rational number. For example, -3 is an integer but it can be thought of as the rational number $\frac{-3}{1}$.

But, set Q contains fractions like $\frac{-2}{3}$ and $\frac{1}{4}$ that are rational numbers, but are not natural numbers, whole numbers, or integers.

Is 7 a natural no.? _____ a whole no.? _____ an integer? _____
a rational no.? yes

Is -5 a natural no.? _____ an integer? yes
a rational number? _____

Is $\frac{1}{3}$ a whole no.? no an integer? _____
a rational no.? _____

Is $0.\overline{6}$ a whole no.? _____ an integer? _____ a rational no.? _____

5. Numbers which cannot be expressed in the form $\frac{a}{b}$, where a and b are integers ($b \neq 0$), belong to $\overline{Q}$, the set of irrational numbers.

For example,

MEMORIZE

a. Square roots of numbers that are not perfect squares are irrational.
$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, etc. are irrational.

b. The number π is irrational.

c. All non-periodic decimals are irrational.
6.010 010 001..., 0.697384... are irrational.

Is $\sqrt{12}$ a natural no.? _____ an integer? no a rational no.? _____
an irrational no? _____

$(\sqrt{9}=3)$ Is $\sqrt{9}$ a natural no.? yes a rational no.? _____ an irrational no.? _____

Is 2.030 03... a natural no.? _____ an integer? _____
a rational no.? _____ an irrational no.? _____

6. The set of <u>real numbers</u>, R, is made up of rational numbers and irrational numbers. That is, all the numbers we have discussed in parts 1-5 on pages 26 and 27 are real numbers. Sets N, W, I, Q, and $\overline{Q}$ are all proper subsets of set R. This means that every natural number, whole number, integer, rational number and irrational number is also a real number. The only numbers that you have encountered that do <u>not</u> belong to set R are square roots of negative numbers. For example, $\sqrt{-6}$, $\sqrt{\frac{-2}{3}}$, $\sqrt{-0.9}$ are not real numbers.

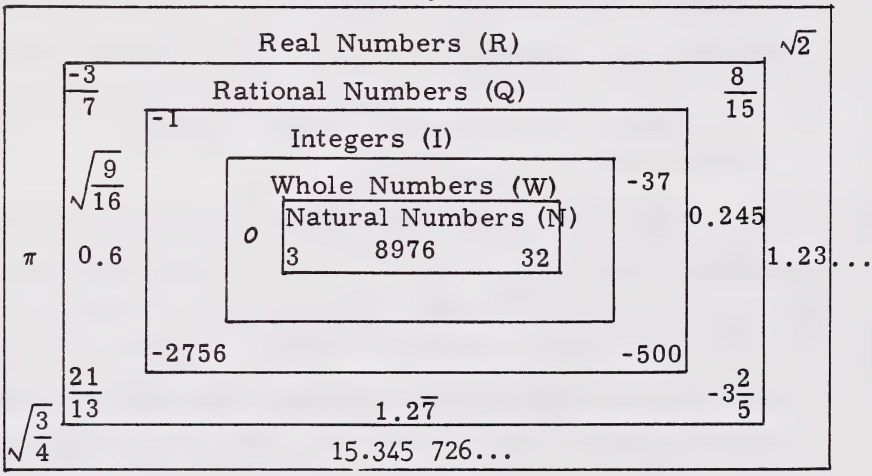
Is 0 a real number? _____ Is -1? _____ Is 236? _____

Is $\frac{1}{2}$? _____ Is $\frac{-1}{2}$? yes Is $3\frac{1}{7}$? _____ Is π? _____ Is $\sqrt{3}$? _____

Is $-\sqrt{3}$? _____ Is $\sqrt{-3}$? _____ Is 1.3? _____ Is 1.3674...? _____

Is $-\sqrt{-1}$? _____

The following diagram illustrates the relationship between sets N, W, I, Q and R. (Note that $N \subset W \subset I \subset Q \subset R$.) SYMBOL FOR "IS A SUBSET OF"



<u>Self-correcting Exercise #5</u>

Answers to this exercise may be found on page 53 of this lesson.

1. Decide if each of the following statements is true or false.

True or False?

(a) 0 is a natural number. __________

(b) 12 is a rational number. __________

(c) $\sqrt{36}$ is a whole number. __________

True or False?

(d) $\sqrt{2}$ is a rational number. ______

(e) -2 is a whole number. ______

(f) -6 is an integer. ______

(g) -6 is a rational number. ______

(h) $\frac{1}{2}$ is an irrational number. ______

(i) $1.\overline{3}$ is a rational number. ______

(j) $\sqrt{12}$ is an irrational number. ______

(k) $\sqrt{-3}$ is a real number. ______

(l) 1.037 is a rational number. ______

(m) 10.035 624... is a rational number. ______

(n) π is a real number. ______

2. N represents the set of natural numbers, W the set of whole numbers, I the set of integers, Q the set of rational numbers, and R the set of real numbers. Give the set or sets to which each of the following numerals belongs. (Remember that $N<W<I<Q<R$.) Thus, if a number belongs to N, it must also belong to sets W, I, Q, and R.)

SYMBOL FOR "IS A SUBSET OF"

	Numeral	Set(s) To Which Numeral Belongs
(a)	13	N, W, I, Q, R
(b)	$\frac{7}{9}$	
(c)	0	
(d)	π	
(e)	$\sqrt{-8}$	
(f)	$\sqrt{4}$	
(g)	-11	
(h)	$\sqrt{6}$	
(i)	$1.2\overline{4}$	
(j)	2.010 01...	

EXERCISE - The Real Number System

1. Fill in the blanks.

   (a) The ____________________ property tells us that between any two real numbers, there is another real number.

   (b) The ____________________ property tells us that every real number corresponds to a point on a solid number line.

   (c) The set of real numbers is the union of the set of ____________ numbers and the set of ____________________ numbers.

   (d) The set of rational numbers is a proper subset of the set of ____________________ numbers.

   (e) The set of integers is a proper subset of the set of __________________ numbers and the set of __________________ numbers.

   (f) The set of rational numbers and the set of __________________ numbers are disjoint sets.

   (g) The decimal 0.363 663 666... is an example of a ________________ decimal.

   (h) The number 1.732 is a ____________________ approximation of the ____________________ number $\sqrt{3}$.

   (i) $\sqrt{10} > -\sqrt{10}$ since $\sqrt{10}$ lies to the ____________________ of $-\sqrt{10}$ on the number line.

   (j) Another name for the set of non-periodic decimals is the set of ____________________ numbers.

   (k) The ____________________ number system and the ______________ number system form fields.

   (l) $3 + \sqrt{2} = \sqrt{2} + 3$ because the operation of ______________________ is ____________________ in set R.

(m) The additive inverse of $\sqrt{10}$ is ____________________.

(n) The multiplicative identity in set R is ____________________.

(o) Every ____________ real number has two square roots that are real numbers. These square roots are equal in ______________ value but opposite in sign.

(p) $\sqrt{7}$ represents the ________________ square root of 7, while $-\sqrt{7}$ represents the ________________ square root of 7.

(q) The number 121 is a ________________ square because $\sqrt{121}$ = __________ which is a rational number.

(r) The point corresponding to the irrational number -2.6438... lies between the integers _____ and _____ on the number line.

2. Insert the symbol "<", ">", or "=" between each of the following pairs of numbers.

(a)	-7 _____ 2		(b)	-9 _____ -12	
(c)	$\frac{-8}{2}$ _____ -4		(d)	$\sqrt{10}$ _____ 3	
(e)	$\frac{4}{5}$ _____ 0.8		(f)	$\sqrt{5}$ _____ $-\sqrt{5}$	
(g)	$\frac{2}{3}$ _____ $0.\overline{6}$		(h)	$\frac{2}{3}$ _____ $\frac{5}{6}$	
(i)	$\frac{-5}{7}$ _____ $\frac{-3}{5}$		(j)	$\frac{1}{4}$ _____ $0.\overline{25}$	
(k)	0.512 _____ $\frac{1}{2}$		(l)	$-\sqrt{5}$ _____ $-\sqrt{6}$	
(m)	$\sqrt{68}$ _____ $8\frac{1}{4}$		(n)	$\sqrt{6}$ _____ $2.\overline{4}$	
(o)	π _____ 3.1562		(p)	$1.106...$ _____ $1.117...$	
(q)	$\frac{5}{11}$ _____ 0.45		(r)	-3.2 _____ -2.1	

3. Fill in the blanks with the missing numbers in order to make each statement true.

(a) $6.\overline{87} \times$ ___ $= 6.\overline{87}$

(b) ___ $+ (-\sqrt{2}) = 0$

(c) $5 \times \pi = \pi \times$ ___

(d) $\frac{1}{\sqrt{5}} \times \sqrt{5} =$ ___

(e) $0 \times \sqrt{11} =$ ___

(f) $\pi +$ ___ $= \pi$

(g) $3(7 + \sqrt{8}) = (3 \times 7) + (3 \times$ ___$)$

(h) $6 + (8 + \sqrt{3}) = (6 + 8) +$ ___

(i) $\sqrt{2} + \sqrt{3} = \sqrt{3} +$ ___

(j) $1\frac{1}{2} \times$ ___ $= 1$

4. Put check marks in the appropriate columns to indicate which sets each number belongs to. (For each number, if you check a given box, you must also check every box to the right since $N \subseteq W \subseteq I \subseteq Q \subseteq R$.)

	N	W	I	Q	R
(a) $\frac{-2}{7}$				✓	✓
(b) 0					
(c) 16					
(d) $5.\overline{3}$					
(e) $\sqrt{11}$					
(f) $-\pi$					
(g) $-\sqrt{4}$					
(h) $\sqrt{-4}$					
(i) 8.6143...					
(j) -1					
(k) 12.35					
(l) $-8\frac{1}{6}$					

5. Check the properties possessed by each of the indicated number systems. (For each property, if you check a given box, you must check every box on the right since $N \subseteq W \subseteq I \subseteq Q \subseteq R$.)

	N	W	I	Q	R
(a) Closure for addition	✓	✓	✓	✓	✓
(b) Closure for multiplication					
(c) Addition is associative					
(d) Multiplication is associative					
(e) Addition is commutative					
(f) Multiplication is commutative					
(g) Multiplication distributes over addition					
(h) Identity element for addition					
(i) Identity element for multiplication					
(j) Inverse elements under addition					
(k) Inverse elements under multiplication					

6. Put a check mark beside each number below that is irrational. (See #5 on page 27.)

(a) $\sqrt{8}$ ✓ (b) $\sqrt{-23}$ ____ (c) $\frac{-3}{7}$ ____

(d) $\sqrt{\frac{1}{4}}$ ____ (e) $-\pi$ ____ (f) $6.\overline{08}$ ____

(g) $-4\frac{1}{8}$ ____ (h) 1.632... ____ (i) $\sqrt{\frac{1}{7}}$ ____

(j) $\sqrt{\frac{-1}{7}}$ ____ (k) $-\sqrt{81}$ ____ (l) $-\sqrt{2.3}$ ____

7. Put a check mark beside each number below that is an integer. (See #3 on page 26.)

(a) -6 ✓ (b) $2\frac{1}{2}$ ____ (c) $-\sqrt{64}$ ____

(d) $\sqrt{-81}$ ____ (e) $\frac{0}{4}$ ____ (f) $\frac{-4}{8}$ ____

(g) $\frac{-9}{3}$ ____ (h) $\sqrt{\frac{9}{16}}$ ____ (i) 0.3 ____

8. Put a check mark beside each number below that is <u>rational</u>. (See #4 on pages 26 and 27.)

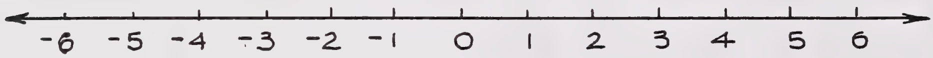
(a) -4	____	(b) $1\frac{1}{6}$	____	(c) 0	____
(d) π	____	(e) 3.51	____	(f) $0.121\ 12\ldots$	____
(g) $0.\overline{7}$	____	(h) $\sqrt{26}$	____	(i) $\sqrt{\frac{4}{81}}$	____
(j) $-\sqrt{0.09}$	____	(k) $\sqrt{3.15}$	____	(l) 133	____

9. Specify each set using set-builder notation and then graph it on the number line provided. (See pages 23 - 25.)

(a) The set of real numbers between -5 and 5 inclusive.

Set-Builder Notation: ______________________________

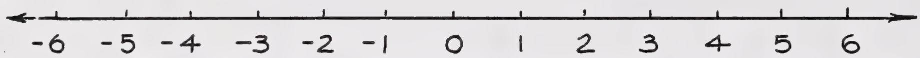
Graph:



(b) The set of all real numbers greater than -2.

Set-Builder Notation: ______________________________

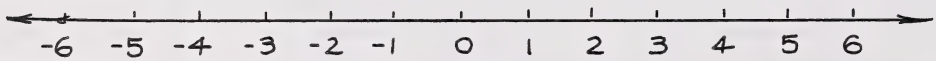
Graph:



(c) The set of all real numbers greater than or equal to -2 and less than 3.

Set-Builder Notation: ______________________________

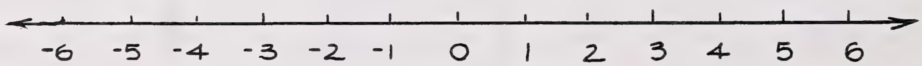
Graph:



(d) The set of all real numbers less than -4 <u>or</u> greater than 1.

Set-Builder Notation: ______________________________

Graph:



Topic Three: Operating With Second Order Radicals

You will recall that $\sqrt{2}$ and $-\sqrt{2}$ are irrational numbers that represent the principal square root of 2 and the negative square root of 2 respectively. Any expression involving a root sign is called a RADICAL. The number under the root sign is called the RADICAND. $\sqrt{2}$ and $-\sqrt{2}$ are radicals with a radicand of 2. More specifically, they are SECOND ORDER RADICALS because they involve square root.

Give examples of three other second order radicals. State the radicand in each case. (Remember that the radicand must always be a positive real number since the square root of a negative number is not real.)

1. $\sqrt{0.16}$ is a second order radical with a radicand of 0.16.
2. $-\sqrt{\frac{3}{4}}$ is a second order radical with a radicand of ______.
3. ______ is a second order radical with a radicand of ______.

A MIXED RADICAL involves the product of a rational number and a radical. For example, $2\sqrt{5}$ is a mixed radical.

RATIONAL NUMBER RADICAL

$$\underbrace{2\sqrt{5}}_{\text{MIXED RADICAL}}$$

Give examples of five other mixed radicals.

$\frac{1}{2}\sqrt{2}$, ______, ______, ______, ______

A. Product of Second Order Radicals

The product of two radicals of the second order can be written as a single radical of the second order.

EXAMPLE: Write the product $\sqrt{4} \times \sqrt{9}$ as a single radical.

Solution

$$\begin{aligned} \sqrt{4} \times \sqrt{9} &= 2 \times 3 \\ &= 6 \\ &= \sqrt{36} \end{aligned}$$

The same result could have been obtained by retaining the square root sign and finding the product of the radicands.

$$\text{i.e.} \quad \sqrt{4} \times \sqrt{9} = \sqrt{4 \times 9} \longleftarrow \text{Product of radicands}$$

$$= \sqrt{36}$$

PRODUCT OF SECOND ORDER RADICALS

If $\sqrt{a}$ and $\sqrt{b}$ are square roots which represent real numbers (i.e. $a \geqq 0$ and $b \geqq 0$),

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

This means that the product of two square roots can be found by multiplying the two radicands and placing this product under a square root sign.

EXAMPLES:

PRODUCT OF RADICANDS

$\sqrt{3} \times \sqrt{7} = \sqrt{3 \times 7} = \sqrt{21}$

$\sqrt{5} \times \sqrt{2} = \sqrt{5 \times 2} = \sqrt{10}$

Find the following products by filling in the blanks below.

1. $\sqrt{6} \times \sqrt{11} = \sqrt{\underline{\ 6\ } \times \underline{\ 11\ }} = \sqrt{\underline{\qquad}}$
2. $\sqrt{2} \times \sqrt{15} = \sqrt{\underline{\quad} \times \underline{\quad}} = \sqrt{\underline{\qquad}}$
3. $\sqrt{5} \times \sqrt{3} = \sqrt{\underline{\quad} \times \underline{\quad}} = \sqrt{\underline{\qquad}}$
4. $\sqrt{7} \times \sqrt{10} = \sqrt{\underline{\quad} \times \underline{\quad}} = \sqrt{\underline{\qquad}}$

In some cases, at least one of the two radicals in a product will be negative. Then you must apply the rules for multiplying integers. Remember that:

1. The product of two numbers with like signs is <u>positive</u>.
2. The product of two numbers with unlike signs is <u>negative</u>.

EXAMPLES:

$-\sqrt{7} \times \sqrt{13}$

Since one factor is negative, the product will be <u>negative</u>.

NEGATIVE PRODUCT

$= -\sqrt{7 \times 13}$

$= -\sqrt{91}$

$-\sqrt{5} \times -\sqrt{6}$

Since both factors are negative, the product will be <u>positive</u>.

POSITIVE PRODUCT

$= +\sqrt{5 \times 6}$

$= \sqrt{30}$

Find each of the following products. Make sure that you attach the proper sign to the product.

1. $\sqrt{2} \times -\sqrt{21} = -\sqrt{2 \times 21} = -\sqrt{42}$
2. $-\sqrt{7} \times -\sqrt{15} =$
3. $-\sqrt{6} \times \sqrt{13} =$
4. $-\sqrt{17} \times -\sqrt{2} =$

An interesting result occurs when you find the product of two identical radicals. Study the following examples.

$$\sqrt{2} \times \sqrt{2} = \sqrt{2 \times 2} = \sqrt{4} = 2$$
$$\sqrt{3} \times \sqrt{3} = \sqrt{3 \times 3} = \sqrt{9} = 3$$
$$\sqrt{5} \times \sqrt{5} = \sqrt{5 \times 5} = \sqrt{25} = 5$$
$$\sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}} = \sqrt{\frac{1}{2} \times \frac{1}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Note that in each case the product is a rational number. This rational number is the same as the radicand of each radical.

PRODUCT OF IDENTICAL RADICALS
If $\sqrt{a}$ is a square root that represents a real number (i.e. $a \geqq 0$), $\sqrt{a} \times \sqrt{a} = a$

The above rule can be used to find the product of two identical radicals in one step. The answer will merely be the number that appears under the root sign in each radical.

Find the following products.

RADICAND AND PRODUCT CORRESPOND

1. $\sqrt{10} \times \sqrt{10} =$ 10
2. $\sqrt{7} \times \sqrt{7} =$ _____
3. $\sqrt{0.1} \times \sqrt{0.1} =$ _____
4. $\sqrt{\frac{1}{8}} \times \sqrt{\frac{1}{8}} =$ _____
5. $\sqrt{\frac{2}{3}} \times \sqrt{\frac{2}{3}} =$ _____
6. $\sqrt{62.4} \times \sqrt{62.4} =$ _____
7. $\sqrt{11} \times \sqrt{11} =$ _____
8. $\sqrt{9} \times \sqrt{9} =$ _____

The rule for multiplying two second order radicals can be extended to cover the product of any number of factors.

i.e. $\sqrt{a} \times \sqrt{b} \times \sqrt{c} \times \sqrt{d}\ldots = \sqrt{a \times b \times c \times d \ldots}$

In attaching a sign to the product, one must count the number of negative factors that appear. Remember that:

1. A product is <u>positive</u> if there is an <u>even</u> number of negative factors.

2. A product is <u>negative</u> if there is an <u>odd</u> number of negative factors.

EXAMPLES:

$-\sqrt{2} \times -\sqrt{5} \times -\sqrt{7}$

There is an <u>odd</u> number of negative factors (3). Thus, the product will be <u>negative</u>.

$= -\sqrt{2 \times 5 \times 7}$

$= -\sqrt{70}$

$-\sqrt{2} \times -\sqrt{3} \times \sqrt{5} \times -\sqrt{7} \times -\sqrt{11}$

There is an <u>even</u> number of negative factors (4). Thus, the product will be <u>positive</u>.

$= +\sqrt{2 \times 3 \times 5 \times 7 \times 11}$

$= \sqrt{2310}$

Find the following products. Make sure you attach the proper sign to each product.

1. $\sqrt{5} \times \sqrt{6} \times -\sqrt{7} =$

2. $-\sqrt{\frac{3}{4}} \times -\sqrt{2} \times \sqrt{6} =$

3. $-\sqrt{0.1} \times -\sqrt{6} \times -\sqrt{7} =$

4. $-\sqrt{2} \times -\sqrt{0.1} \times -\sqrt{3} \times -\sqrt{0.2} \times -\sqrt{5} =$

The product of two <u>mixed radicals</u> of the second order can be written as a single mixed radical. This can be done by using the commutative and associative properties of multiplication to group the two rational numbers together and the two radicals together. The rule for multiplying rational numbers can be used to find the product of the two rational numbers. The rule for multiplying radicals can be used to find the product of the two radicals.

EXAMPLE: $\frac{2}{3}\sqrt{5} \times \frac{-3}{4}\sqrt{7} = \boxed{}$

<u>Solution</u>

$$\frac{2}{3}\sqrt{5} \times \frac{-3}{4}\sqrt{7} = \frac{2}{3} \times \sqrt{5} \times \frac{-3}{4} \times \sqrt{7}$$

Use the commutative and associative properties of multiplication to group $\frac{2}{3}$ and $\frac{-3}{4}$ together and $\sqrt{5}$ and $\sqrt{7}$ together.

$$= \left(\frac{2}{3} \times \frac{-3}{4}\right) \left(\sqrt{5} \times \sqrt{7}\right)$$

$$= \left(\frac{\overset{1}{\cancel{2}} \times \overset{-1}{\cancel{-3}}}{\underset{1}{\cancel{3}} \times \underset{2}{\cancel{4}}}\right)\left(\sqrt{5 \times 7}\right)$$

$$= \frac{-1}{2}\sqrt{35}$$

PRODUCT OF SECOND ORDER MIXED RADICALS
If $c\sqrt{a}$ and $d\sqrt{b}$ are mixed radicals which represent real numbers, $c\sqrt{a} \times d\sqrt{b} = cd\sqrt{ab}$

This means that the product of mixed radicals can be found by finding the product of the rational parts and multiplying this by the product of the radicals.

EXAMPLES:

$6\sqrt{5} \times -3\sqrt{2} = (6 \times -3)\sqrt{5 \times 2} = \underline{\qquad\qquad}$

$-5\sqrt{2} \times -8\sqrt{3} = (-5 \times \underline{\quad})\sqrt{2 \times 3} = \underline{\qquad\qquad}$

$\frac{2}{3}\sqrt{11} \times \frac{1}{2}\sqrt{7} = \left(\frac{}{} \underline{\quad} \times \frac{1}{2}\right) \sqrt{11 \times \underline{\quad}} = \underline{\qquad\qquad}$

The rule for multiplying mixed radicals can be extended to cover the product of any number of factors. For example,

$$3\sqrt{2} \times -4\sqrt{5} \times \frac{1}{2}\sqrt{7} = \left(3 \times -4 \times \frac{1}{2}\right)(\sqrt{2 \times 5 \times 7})$$
$$= -6\sqrt{70}$$

Similarly,

$$-2\sqrt{2} \times 5\sqrt{3} \times 6\sqrt{5} = (\underline{\quad} \times \underline{\quad} \times \underline{\quad})\sqrt{\underline{\quad} \times \underline{\quad} \times \underline{\quad}}$$

$$= \underline{\quad}\sqrt{\underline{\quad}}$$

Self-correcting Exercise #6

Answers may be found on page 54 of this lesson.

1. Find the following products.

(a) $\sqrt{3} \times \sqrt{7} =$ ______

(b) $-\sqrt{2} \times -\sqrt{19} =$ ______

(c) $\sqrt{6} \times -\sqrt{17} =$ ______

(d) $\sqrt{30} \times \sqrt{30} =$ ______

(e) $-\sqrt{\frac{1}{3}} \times \sqrt{\frac{1}{6}} =$ ______

(f) $-\sqrt{5} \times \sqrt{5} =$ ______

(g) $-\sqrt{\frac{1}{3}} \times -\sqrt{\frac{1}{3}} =$ ______

(h) $2\sqrt{10} \times 8\sqrt{3} =$ ______

(i) $-3\frac{1}{2}\sqrt{2} \times 2\sqrt{3} =$ ______

(j) $\frac{-1}{2}\sqrt{13} \times \frac{1}{3}\sqrt{3} =$ ______

(k) $\sqrt{3} \times \sqrt{\frac{1}{4}} \times \sqrt{8} =$ ______

(l) $9\sqrt{2} \times 5\sqrt{3} \times -7\sqrt{5} =$ ______

(m) $3\sqrt{3.5} \times -4\sqrt{0.2} \times -2\sqrt{3} =$ ______

B. Quotient of Second Order Radicals

The quotient of two radicals of the second order can be written as a single radical of the second order.

EXAMPLE: Write the quotient $\frac{\sqrt{16}}{\sqrt{4}}$ as a single radical.

Solution

$$\frac{\sqrt{16}}{\sqrt{4}} = \frac{4}{2}$$
$$= 2$$
$$= \sqrt{4}$$

The same result could have been obtained by retaining the radical sign and finding the quotient of the radicands.

i.e. $\frac{\sqrt{16}}{\sqrt{4}} = \sqrt{\frac{16}{4}} = \sqrt{4}$

quotient of radicands

QUOTIENT OF SECOND ORDER RADICALS

If $\sqrt{a}$ and $\sqrt{b}$ are square roots which represent real numbers,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

This means that the quotient of two square roots can be found by dividing the two radicands and placing this quotient under a square root sign.

EXAMPLES:

QUOTIENT OF RADICANDS

$$\frac{\sqrt{14}}{\sqrt{2}} = \sqrt{\frac{14}{2}} = \sqrt{7}$$

$$\frac{-\sqrt{15}}{\sqrt{3}} = -\sqrt{\frac{15}{3}} = -\sqrt{5}$$

QUOTIENT IS NEGATIVE SINCE NUMERATOR IS NEGATIVE.

Find the following quotients by filling in the blanks below.

1. $\dfrac{-\sqrt{77}}{-\sqrt{7}} = \sqrt{\dfrac{77}{\square}} = \sqrt{\_\_\_}$

2. $\dfrac{\sqrt{182}}{\sqrt{13}} = \sqrt{\dfrac{\square}{\square}} = \sqrt{\_\_\_}$

3. $\dfrac{-\sqrt{110}}{\sqrt{22}} = -\sqrt{\dfrac{\square}{\square}} = -\sqrt{\_\_\_}$

The quotient of two mixed radicals can be written as a single mixed radical. This can be done by finding the quotient of the two rational parts and multiplying this by the quotient of the two radicals.

EXAMPLE: $\dfrac{-10\sqrt{30}}{15\sqrt{2}} = \boxed{}$

Solution

QUOTIENT OF RADICALS

$$\frac{-10\sqrt{30}}{15\sqrt{2}} = \frac{-10}{15} \times \frac{\sqrt{30}}{\sqrt{2}}$$

QUOTIENT OF RATIONAL NUMBERS

$$= \frac{-2}{3}\sqrt{\frac{30}{2}}$$

$$= \frac{-2}{3}\sqrt{15}$$

QUOTIENT OF SECOND ORDER MIXED RADICALS

If $c\sqrt{a}$ and $d\sqrt{b}$ are mixed radicals which represent real numbers,

$$\frac{c\sqrt{a}}{d\sqrt{b}} = \frac{c}{d}\sqrt{\frac{a}{b}}$$

Find the following quotients by filling in the blanks below.

1. $\dfrac{-9\sqrt{6}}{3\sqrt{2}} = \dfrac{-9}{\_\_} \times \dfrac{\sqrt{6}}{\sqrt{\_\_}} = \_\_\sqrt{\_\_}$

2. $\dfrac{12\sqrt{95}}{16\sqrt{5}} = \dfrac{\_\_}{16} \times \sqrt{\dfrac{\_\_}{\_\_}} = \_\_\sqrt{\_\_}$

3. $\dfrac{-14\sqrt{2}}{-2\sqrt{2}} =$

C. <u>Sum or Difference of Second Order Mixed Radicals</u>

Mixed radicals of the second order are said to be LIKE TERMS if they have the same radicand. For example, $5\sqrt{3}$ and $\frac{-7}{3}\sqrt{3}$ are like terms since they both have a radicand of 3.

By using the distributive property in reverse, we can write the sum or difference of like mixed radicals of the second order as a single mixed radical of the second order.

EXAMPLE: Write the sum $7\sqrt{2} + 3\sqrt{2}$ as a single mixed radical.

<u>Solution</u>

$$\begin{aligned} & 7\sqrt{2} + 3\sqrt{2} \\ &= (7 + 3)\sqrt{2} \qquad \text{(distributive property in reverse)} \\ &= 10\sqrt{2} \end{aligned}$$

ADDITION OR SUBTRACTION OF LIKE MIXED RADICALS OF THE SECOND ORDER

If $c\sqrt{a}$ and $d\sqrt{a}$ are mixed radicals which represent real numbers,

$$c\sqrt{a} + d\sqrt{a} = (c + d)\sqrt{a}$$
$$c\sqrt{a} - d\sqrt{a} = (c - d)\sqrt{a}$$

This law can be expanded to cover the sum and difference of any number of second order mixed radicals that are like terms. For example,

$$\frac{2}{7}\sqrt{3} - \frac{1}{7}\sqrt{3} + 2\sqrt{3} - \frac{3}{7}\sqrt{3} = \left(\frac{2}{7} - \frac{1}{7} + 2 - \frac{3}{7}\right)\sqrt{3}$$

$$= \left(\frac{2 - 1 + 14 - 3}{7}\right)\sqrt{3}$$

$$= \frac{12}{7}\sqrt{3}$$

Similarly,

$$2\sqrt{5} - 6\sqrt{5} + 9\sqrt{5} - 15\sqrt{5} = (\_\_ - \_\_ + \_\_ - \_\_)\sqrt{5}$$

$$= \_\_\sqrt{5}$$

Self-correcting Exercise #7

Answers may be found on page 54 of this lesson.

1. In each question, state whether the two numbers are "like terms" or "unlike terms".

	Like or Unlike Terms?
(a) $3\sqrt{6}$, $6\sqrt{3}$	________
(b) $5\sqrt{2}$, $\frac{1}{2}\sqrt{2}$	________
(c) $-4\sqrt{3}$, $\sqrt{3}$	________
(d) $\frac{3}{5}\sqrt{5}$, $\frac{2}{5}\sqrt{3}$	________
(e) $-5\sqrt{7}$, $5\sqrt{7}$	________

2. Simplify each of the following sums and differences.

(a) $8\sqrt{3} + 5\sqrt{3} =$

(b) $12\sqrt{5} - 9\sqrt{5} =$

(c) $6\sqrt{2} - 15\sqrt{2} =$

(d) $3\sqrt{10} + 6\sqrt{10} - 9\sqrt{10} =$

D. Simplifying Radicals

If $c\sqrt{a}$ is a mixed radical which represents a real number, it is said to be in SIMPLEST FORM if it satisfies the following conditions:

(1) The radicand "a" contains no factor which has a rational square root.
(2) The radicand "a" is not a fraction.

EXAMPLES:

The number $\sqrt{8}$ is not in simplest form since the radicand 8 contains the factor 4 which has a rational square root of 2.

The number $\sqrt{\frac{3}{7}}$ is not in simplest form since the radicand $\frac{3}{7}$ is a fraction.

The number $4\sqrt{10}$ is in simplest form since the radicand 10 is not a fraction and it does not contain any factor which has a rational square root.

If a radicand contains a factor which has a rational square root, this factor can be taken out from under the root sign.

EXAMPLE: Write the radical $\sqrt{108}$ in simplest form.

Solution

We must look for the largest factor of 108 which has a rational square root. Note that 36 is a factor of 108 and has a rational square root of 6.

$$\sqrt{108} = \sqrt{36 \times 3}$$

If we apply the rule for multiplying radicals, in reverse, we know that $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

i.e.
$$\begin{aligned}\sqrt{108} &= \sqrt{36 \times 3} \\ &= \sqrt{36} \times \sqrt{3} \\ &= 6\sqrt{3}\end{aligned}$$

Thus, $\sqrt{108}$ can be expressed in simplest form as $6\sqrt{3}$.

Simplify each radical below by removing perfect square factors.

1. $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$

2. $\sqrt{300} = \sqrt{100 \times \_\_} = \sqrt{100} \times \sqrt{\_\_} = 10\sqrt{\_\_}$

3. $\sqrt{405} = \sqrt{81 \times __} = \sqrt{__} \times \sqrt{__} = $ ________

You must <u>memorize</u> the squares of the natural numbers from 2 to 12 inclusive since you must use these perfect squares in simplifying radicals. These squares are:

4, 9, ____, 25, 36, ____, ____, 81, 100, ____, 144

When finding sums or differences, you can often discover like terms by first expressing each radical in <u>simplest form</u>. For example, at first glance, the terms in the sum $(2\sqrt{28} + 3\sqrt{63})$ appear to be unlike. But, once the terms are put in simplest form, you will find that they both contain the radical $\sqrt{7}$.

i.e.
$$\begin{aligned} 2\sqrt{28} + 3\sqrt{63} &= 2\sqrt{4 \times 7} + 3\sqrt{9 \times 7} \\ &= 2\sqrt{4}\sqrt{7} + 3\sqrt{9}\sqrt{7} \\ &= (2 \times 2 \times \sqrt{7}) + (3 \times 3 \times \sqrt{7}) \\ &= 4\sqrt{7} + 9\sqrt{7} \\ &= 13\sqrt{7} \end{aligned}$$

Simplify each term in the expression below in order to discover like terms. Then add.

$$\begin{aligned} 2\sqrt{8} + 3\sqrt{50} + \sqrt{128} &= 2\sqrt{\underline{4} \times 2} + 3\sqrt{\underline{\quad} \times 2} + \sqrt{\underline{\quad} \times 2} \\ &= 2\sqrt{\underline{4}}\sqrt{2} + 3\sqrt{\underline{\quad}}\sqrt{2} + \sqrt{\underline{\quad}}\sqrt{2} \\ &= (2 \times \underline{2}\sqrt{2}) + (3 \times \underline{\quad}\sqrt{2}) + (\underline{\quad}\sqrt{2}) \\ &= \underline{4}\sqrt{2} + \underline{\quad}\sqrt{2} + \underline{\quad}\sqrt{2} \\ &= (\underline{4} + \underline{\quad} + \underline{\quad})\sqrt{2} \\ &= \underline{\quad}\sqrt{2} \end{aligned}$$

<u>Self-correcting Exercise #8</u>

Answers may be found on page 55 of this lesson.

1. For each question, state the <u>largest</u> perfect square that will divide into each radicand. (See list of perfect squares on page 45 of this lesson.)

   (a) $\sqrt{243}$ ______ (b) $\sqrt{600}$ ______ (c) $\sqrt{180}$ ______

   (d) $\sqrt{147}$ ______ (e) $\sqrt{96}$ ______ (f) $\sqrt{150}$ ______

2. Simplify each radical by removing perfect square factors.

   (a) $\sqrt{480}$ =

   (b) $\sqrt{50}$ =

   (c) $\sqrt{384}$ =

3. Perform the following operations. All answers should be expressed in simplest form.

(a) $\sqrt{2} - \sqrt{8} =$

(b) $2\sqrt{3} \times \sqrt{2} \times -\sqrt{6} =$

(c) $4\sqrt{90} + 3\sqrt{10} =$

(d) $3\sqrt{3} \times 2\sqrt{2} =$

(e) $\frac{3}{4}\sqrt{5} \times \frac{-2}{3}\sqrt{10} =$

EXERCISE - Operating With Radicals.

1. Fill in the blanks.

(a) When you multiply second order radicals, you retain the square root sign and find the product of the ______________________.

(b) Numbers which can be expressed as the product of a rational number and a radical are called ______________ radicals.

(c) $\sqrt{50}$ is not in simplest form because the radicand contains the factor ________ which has a rational square root of ______.

(d) The sum of second order mixed radicals can be expressed as a single mixed radical only when the radicals have the same ____________.

(e) $\sqrt{\frac{2}{3}}$ is not in simplest form because the radicand $\frac{2}{3}$ is a

____________________.

(f) The product of two ____________________ radicals is always a rational number.

2. Express the following radicals in simplest form. (See Section D, p. 44.)

(a) $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{\quad} \times \sqrt{\quad} =$

(b) $\sqrt{72} = \sqrt{\quad} =$ __________

(c) $\sqrt{80} = \sqrt{\quad \times \quad} =$

(d) $\sqrt{50} =$

(e) $\sqrt{121} =$

(f) $\sqrt{72} =$

(g) $\sqrt{1000} =$

(h) $2\sqrt{500} =$

3. Perform the following operations. Answers should be expressed in simplest form.

See box on p. 37

(a) $\sqrt{7} \times \sqrt{7} =$

(b) $\sqrt{\frac{2}{3}} \times -\sqrt{\frac{2}{3}} =$

(c) $5\sqrt{6} \times -3\sqrt{6} =$

(d) $-3\sqrt{11} \times -6\sqrt{5} = (-3 \times -6)(\sqrt{11} \times \sqrt{5}) =$

(e) $5\sqrt{3} \times 11\sqrt{15}$

(f) $\frac{\sqrt{82}}{\sqrt{2}} = \sqrt{\frac{\quad}{2}} =$

(g) $\frac{-5\sqrt{7}}{3\sqrt{7}} =$

(h) $\frac{2\sqrt{15}}{\sqrt{5}}$

(i) $2\sqrt{7} \times 3\sqrt{7}$

(j) $\frac{6\sqrt{55}}{-2\sqrt{5}} = \frac{6}{-2} \times \sqrt{\frac{\quad}{\quad}} =$

(k) $3\sqrt{2} - 9\sqrt{2} = (3 - 9)\sqrt{2} =$

(l) $8\sqrt{5} - 2\sqrt{5} + \sqrt{5} =$

(m) $\sqrt{24} + \sqrt{54} = \sqrt{4 \times 6} + \sqrt{9 \times 6} =$

(n) $(3\sqrt{2})(5\sqrt{3}) + (8\sqrt{6})(-2) = (3 \times 5)\sqrt{\quad} + (8 \times -2)\sqrt{\quad} =$

(o) $\sqrt{12} + 3\sqrt{27} - 2\sqrt{75} = \sqrt{4 \times 3} + 3\sqrt{\quad \times 3} - 2\sqrt{\quad \times 3} =$

(p) $3\sqrt{72} + 5\sqrt{32} - 6\sqrt{50} = 3\sqrt{36 \times 2} + 5\sqrt{\quad \times 2} - 6\sqrt{\quad \times 2} =$

4. Write each radical in simplest form. Then use the table of square roots at the end of this lesson to find a rational approximation.

(a) $\sqrt{147} = \sqrt{\underline{49} \times \underline{3}}$

$= \sqrt{\underline{49}} \times \sqrt{\underline{3}}$

$= \underline{7}\sqrt{\underline{3}}$

"IS APPROXIMATELY EQUAL TO" $\doteq \underline{7} \times \underline{1.732}$ SEE PAGE 50.

$\doteq$ ________

(b) $\sqrt{325} = \sqrt{\_\_ \times \_\_}$

$= \sqrt{\_\_} \times \sqrt{\_\_}$

$= \_\_\sqrt{\_\_}$

$\doteq \_\_ \times \_\_\_\_$

$\doteq$ ________

(c) $\sqrt{448} = \sqrt{\_\_ \times \_\_}$

$= \sqrt{\_\_} \times \sqrt{\_\_}$

$=$

(d) $\sqrt{242} = \sqrt{\_\_ \times \_\_}$

$= \sqrt{\_\_} \times \sqrt{\_\_}$

$=$

(e) $\sqrt{1100} =$

(f) $\sqrt{212} =$

TABLE OF SQUARE ROOTS

n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$
1	1.000	26	5.099	51	7.141	76	8.718
2	1.414	27	5.196	52	7.211	77	8.775
3	1.732	28	5.292	53	7.280	78	8.832
4	2.000	29	5.385	54	7.348	79	8.888
5	2.236	30	5.477	55	7.416	80	8.944
6	2.449	31	5.568	56	7.483	81	9.000
7	2.646	32	5.657	57	7.550	82	9.055
8	2.828	33	5.745	58	7.616	83	9.110
9	3.000	34	5.831	59	7.681	84	9.165
10	3.162	35	5.916	60	7.746	85	9.220
11	3.317	36	6.000	61	7.810	86	9.274
12	3.464	37	6.083	62	7.874	87	9.327
13	3.606	38	6.164	63	7.937	88	9.381
14	3.742	39	6.245	64	8.000	89	9.434
15	3.873	40	6.325	65	8.062	90	9.487
16	4.000	41	6.403	66	8.124	91	9.539
17	4.123	42	6.481	67	8.185	92	9.592
18	4.243	43	6.557	68	8.246	93	9.644
19	4.359	44	6.633	69	8.307	94	9.695
20	4.472	45	6.708	70	8.367	95	9.747
21	4.583	46	6.782	71	8.426	96	9.798
22	4.690	47	6.856	72	8.485	97	9.849
23	4.796	48	6.928	73	8.544	98	9.899
24	4.899	49	7.000	74	8.602	99	9.950
25	5.000	50	7.071	75	8.660	100	10.000

Key to Self-correcting Exercises in Lesson 6

Exercise #1, page 13

1. (a) $\sqrt{625} = 25$ rational
   (b) $\sqrt{0.1}$ irrational
   (c) $-\sqrt{0.01} = -0.1$ rational
   (d) $\sqrt{19}$ irrational
   (e) $\sqrt{-19}$ neither
   (f) $-\sqrt{81} = -9$ rational

   NEGATIVE NO. UNDER ROOT SIGN.

   (g) $-\sqrt{-81}$ neither
   (h) $\sqrt{0} = 0$ rational
   (i) $-\sqrt{1} = -1$ rational
   (j) $\sqrt{-1}$ neither
   (k) $\sqrt{\frac{3}{4}}$ irrational
   (l) $\sqrt{\frac{9}{4}} = \frac{3}{2}$ rational

2. (a) $\sqrt{81} = 9$ or $-\sqrt{81} = -9$
   (b) $\sqrt{7}$ or $-\sqrt{7}$
   } All positive numbers have two square roots.
   (c) $\sqrt{\frac{1}{4}} = \frac{1}{2}$ ($\sqrt{a}$ represents the principal square root of a.)
   (d) $-\sqrt{64} = -8$ ($-\sqrt{a}$ represents the negative square root of a.)
   (e) $\sqrt{19}$ ($\sqrt{a}$ represents the principal square root of a.)

Exercise #2, page 20

1. (a) real (rational); $3.\overline{7}$ is equivalent to the rational number $\frac{34}{9}$.
   (b) real (irrational); Non-periodic decimals are irrational.
   (c) not real; Square roots of negative numbers are not real.
   (d) real (rational); All integers are rational numbers.
   (e) real (rational); 0.16 is a perfect square.
   (f) real (rational); $\frac{-7}{8}$ is in the rational form $\frac{a}{b}$ where $a, b, \in I$.
   (g) real (irrational); π is equivalent to the non-periodic decimal 3.141 592 6... .
   (h) real (rational); 4.3207 is equivalent to the rational number $\frac{43\ 200}{10\ 000}$
   (i) not real; Square roots of negative numbers are not real.
   (j) real (irrational); 0.465 is <u>not</u> a perfect square.
   (k) real (rational); $5\frac{7}{8}$ is equivalent to the rational number $\frac{47}{8}$.

Exercise #3, page 22

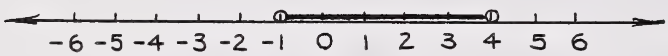
1.

	Real Numbers		Decimal Values		Larger Number
(a)	$1\frac{1}{2}$,	$\sqrt{2}$	1.5,	1.414	$1\frac{1}{2}$ (since 1.5 > 1.414)
(b)	$-\sqrt{5}$,	$-\sqrt{6}$	-2.236,	-2.449	$-\sqrt{5}$ (since -2.236 > -2.449)
(c)	π,	$\sqrt{10}$	3.142,	3.162	$\sqrt{10}$ (since 3.162 > 3.142)
(d)	$\frac{15}{11}$,	$\sqrt{3}$	$1.\overline{36}$,	1.732	$\sqrt{3}$ (since 1.732 > 1.36)
(e)	$\sqrt{62}$,	$\frac{63}{8}$	7.874,	7.875	$\frac{63}{8}$ (since 7.875 > 7.874)

Exercise #4, page 24

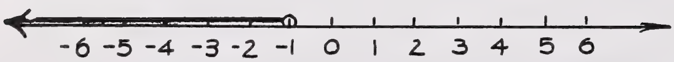
1. (a) $\{x \mid -1 < x < 4,\ x \in R\}$

(Use hollow dots at -1 and 4 since these values are not included.)



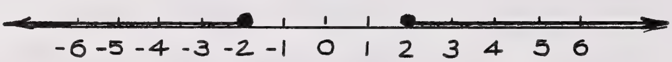
(b) $\{x \mid x < -1,\ x \in R\}$

(Put a hollow dot at -1 since it is not included. The graph extends indefinitely to the left.)



(c) $\{x \mid x \leqq -2 \text{ or } x \geqq 2,\ x \in R\}$

(Put solid dots at -2 and 2 since these values are included. The graph consists of all real numbers that lie to the left of -2 and to the right of 2.)



Exercise #5, page 28

1. (a) false; 0 does not belong to the set N = $\{1, 2, 3, \ldots\}$.
   (b) true; 12 can be written in the form $\frac{12}{1}$.
   (c) true; $\sqrt{36} = 6$ which is a whole number.
   (d) false; 2 is not a perfect square, so $\sqrt{2}$ is irrational.
   (e) false; -2 does not belong to the set W = $\{0, 1, 2, 3, \ldots\}$.
   (f) true; -6 belongs to the set I = $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
   (g) true; -6 can be written in the form $\frac{-6}{1}$.
   (h) false; $\frac{1}{2}$ is rational. (It is written in the form $\frac{a}{b}$ where a and b are integers.)
   (i) true; $1.\overline{3}$ is equivalent to the rational number $\frac{4}{3}$.
   (j) true; $\sqrt{12}$ is irrational since 12 is not a perfect square.
   (k) false; Square roots of negative numbers are not real.
   (l) true; 1.037 is equivalent to the rational number $\frac{1037}{1000}$.
   (m) false; Non-periodic decimals are not rational numbers.
   (n) true; All irrational numbers are real numbers.

SYMBOL FOR "IS A SUBSET OF"

2. (a)	13;	N, W, I, Q, R	(13 is a natural number. $N \subset W \subset I \subset Q \subset R$)
(b)	$\frac{7}{9}$;	Q, R	($\frac{7}{9}$ is a rational number. $Q \subset R$)
(c)	0;	W, I, Q, R	(0 is a whole number. $W \subset I \subset Q \subset R$)
(d)	π	R	(All irrational numbers belong to R.)
(e)	$\sqrt{-8}$;	none	(Square roots of negative numbers aren't real.)
(f)	$\sqrt{4}$;	N, W, I, Q, R	(2 is a natural number. $N \subset W \subset I \subset Q \subset R$)
(g)	-11;	I, Q, R	(-11 is an integer. $I \subset Q \subset R$)
(h)	$\sqrt{6}$;	R	(All irrational numbers belong to R.)
(i)	$1.\overline{24}$;	Q, R	($1.\overline{24}$ is a rational number. $Q \subset R$)
(j)	2.010 01... ;	R	(All irrational numbers belong to R.)

Exercise #6, page 40

1. (a) $\sqrt{3} \times \sqrt{7} = \sqrt{21}$ (b) $\sqrt{2} \times -\sqrt{19} = \sqrt{38}$

(c) $\sqrt{6} \times -\sqrt{17} = -\sqrt{102}$ (d) $\sqrt{30} \times \sqrt{30} = 30$

(e) $-\sqrt{\frac{1}{3}} \times \sqrt{\frac{1}{6}} = -\sqrt{\frac{1}{18}}$ (f) $-\sqrt{5} \times \sqrt{5} = -5$

(g) $-\sqrt{\frac{1}{3}} \times -\sqrt{\frac{1}{3}} = \frac{1}{3}$ (h) $2\sqrt{10} \times 8\sqrt{3} = 16\sqrt{30}$

(i) $-3\frac{1}{2}\sqrt{2} \times 2\sqrt{3} = -7\sqrt{6}$ (j) $\frac{-1}{2}\sqrt{13} \times \frac{1}{3}\sqrt{3} = \frac{-1}{6}\sqrt{39}$

(k) $\sqrt{3} \times \sqrt{\frac{1}{4}} \times \sqrt{8} = \sqrt{3 \times \frac{1}{4} \times 8} = \sqrt{6}$

(l) $9\sqrt{2} \times 5\sqrt{3} \times -7\sqrt{5} = (9 \times 5 \times -7)(\sqrt{2 \times 3 \times 5}) = -315\sqrt{30}$

(m) $3\sqrt{3.5} \times -4\sqrt{0.2} \times -2\sqrt{3} = (3 \times -4 \times -2)\sqrt{3.5 \times 0.2 \times 3} = 24\sqrt{2.1}$

Exercise 7, page 43

1. (a) unlike (b) like (c) like (d) unlike (e) like

2. (a) $8\sqrt{3} + 5\sqrt{3} = (8 + 5)\sqrt{3} = 13\sqrt{3}$

(b) $12\sqrt{5} - 9\sqrt{5} = (12 - 9)\sqrt{5} = 3\sqrt{5}$

(c) $6\sqrt{2} - 15\sqrt{2} = (6 - 15)\sqrt{2} = -9\sqrt{2}$

(d) $3\sqrt{10} + 6\sqrt{10} - 9\sqrt{10} = (3 + 6 - 9)\sqrt{10} = 0\sqrt{10} = 0$

Exercise #8, page 46

1. (a) 81 (b) 100 (c) 36

(d) 49 (e) 16 (f) 25

2. (a) $\sqrt{480} = \sqrt{16 \times 30} = \sqrt{16} \times \sqrt{30} = 4\sqrt{30}$

(b) $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$

(c) $\sqrt{384} = \sqrt{64 \times 6} = \sqrt{64} \times \sqrt{6} = 8\sqrt{6}$

3. (a) $\sqrt{2} - \sqrt{8} = \sqrt{2} - \sqrt{4 \times 2} = \sqrt{2} - 2\sqrt{2} = (1 - 2)\sqrt{2} = -\sqrt{2}$

(b) $2\sqrt{3} \times \sqrt{2} \times -\sqrt{6} = -2\sqrt{36} = -2 \times 6 = -12$

(c) $4\sqrt{90} + 3\sqrt{10} = 4\sqrt{9 \times 10} + 3\sqrt{10} = 12\sqrt{10} + 3\sqrt{10} = (12 + 3)\sqrt{10} = 15\sqrt{10}$

(d) $3\sqrt{3} \times 2\sqrt{2} = 3 \times 2 \times \sqrt{3 \times 2} = 6\sqrt{6}$

(e) $\frac{3}{4}\sqrt{5} \times \frac{-2}{3}\sqrt{10} = \frac{3 \times -2}{4 \times 3}\sqrt{50} = \frac{-1}{2}\sqrt{25 \times 2} = \frac{-5}{2}\sqrt{2}$

End of Lesson 6